GROUP FIELD THEORY:
MODERN PERSPECTIVES

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SUMMARY

- Simple Example and motivation
- Concrete results
- Modern Avenues of Research
- Conclusion

SIMPLE EXAMPLE

3d Group field theory - Boulatov Model

Field $\Phi:\ \mathcal{G}^3 \to \mathbb{C}$ $\Phi(g_1, g_2, g_3)$

Symmetries

Right Shift Symmetry

$$\Phi(g_1\bar{g}, g_2\bar{g}, g_3\bar{g}) = \Phi(g_1, g_2, g_3) \quad \text{for all} \quad \bar{g} \in \mathcal{G}$$

Impose using a projector

$$\Phi(g_1, g_2, g_3) = \int_{\mathcal{G}} dg\ \phi(g_1 g, g_2 g, g_3 g) \equiv\ P_g \phi$$

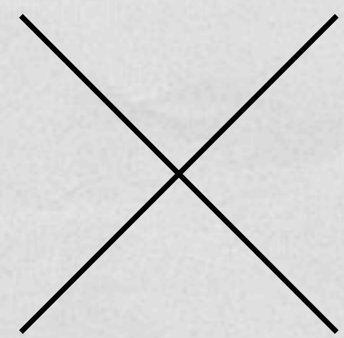
Permutation Symmetry

$$\Phi(g_1, g_2, g_3) = \Phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}) \quad \text{for all} \quad \sigma \in S_n$$
$$\Phi^\dagger(g_1, g_2, g_3) = \Phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}) \quad \text{for all} \quad \sigma \in A_n$$

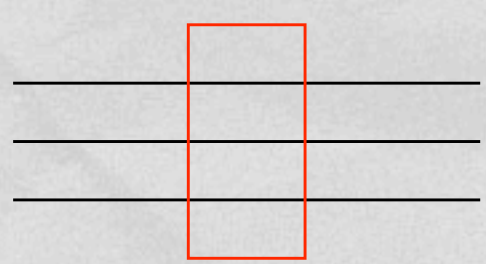
3d Group field theory - Boulatov Model

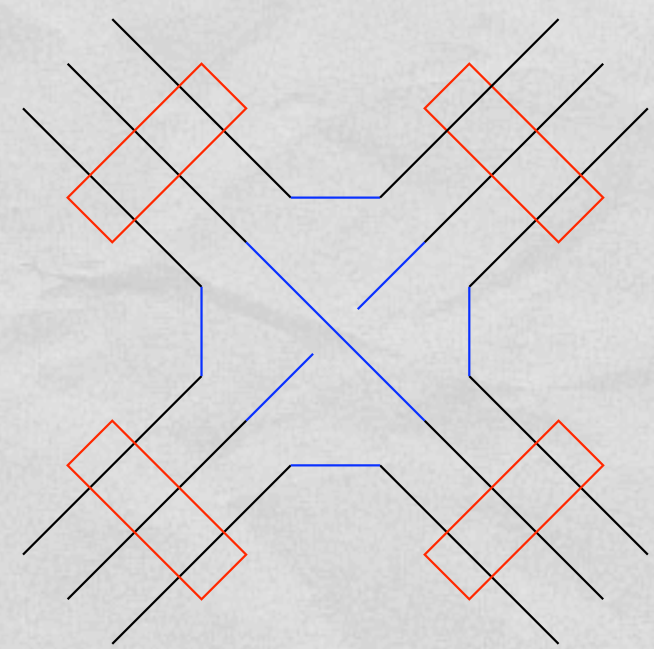
Action

$$\mathcal{S}[\Phi,\Phi^{\dagger}] \;=\; \frac{1}{2}\int_{\mathcal{G}^3} |\Phi|^2 \quad + \quad \frac{\lambda}{4!}\int_{\mathcal{G}^6} \Phi^4$$



$$\mathcal{S}[\Phi,\Phi^{\dagger}] \;=\; \frac{1}{2}\int_{\mathcal{G}^3} |\Phi|^2(g_1,g_2,g_3) \;+\; \frac{\lambda}{4!}\int_{\mathcal{G}^6} \Phi(g_1,g_2,g_3)\Phi(g_4,g_5,g_3)\Phi(g_4,g_2,g_6)\Phi(g_1,g_5,g_6)$$





3d Group field theory - Boulatov Model

Momentum Space

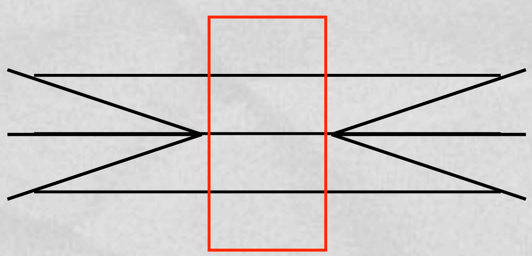
Field $\mathcal{G} = SU(2)$

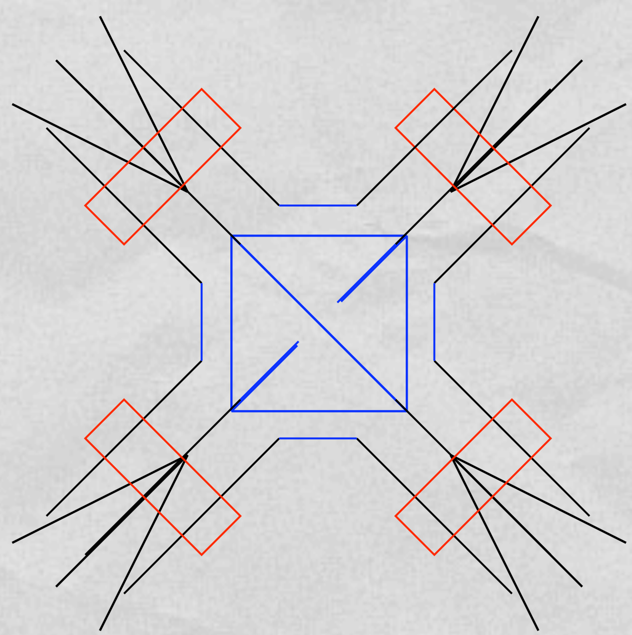
$$\Phi(g_1, g_2, g_3) = P_g\phi(g_1, g_2, g_3) = \sum \phi^{j_1 j_2 j_3}_{m_1 m_2 m_3} C^{j_1 j_2 j_3}_{n_1 n_2 n_3} \quad \times \quad D^{j_1}_{m_1 n_1}(g_1) D^{j_2}_{m_2 n_2}(g_2) D^{j_3}_{m_3 n_3}(g_3)$$

modes *akin to plane waves*

Action

$$\mathcal{S}[\phi, \phi^\dagger] \quad = \quad \frac{1}{2} \sum |\phi|^2 \quad + \quad \frac{\lambda}{4!} \sum \phi^4 \times \{6j - symbol\}$$

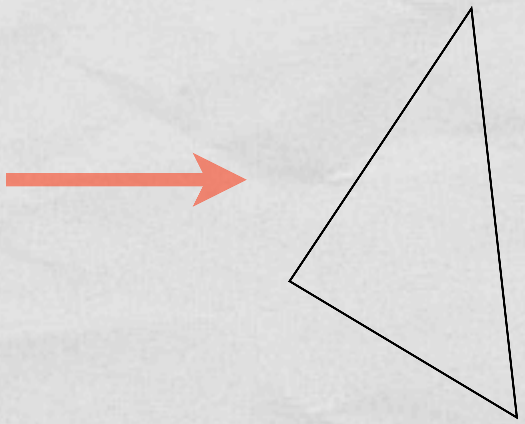




3d Group field theory - Boulatov Model

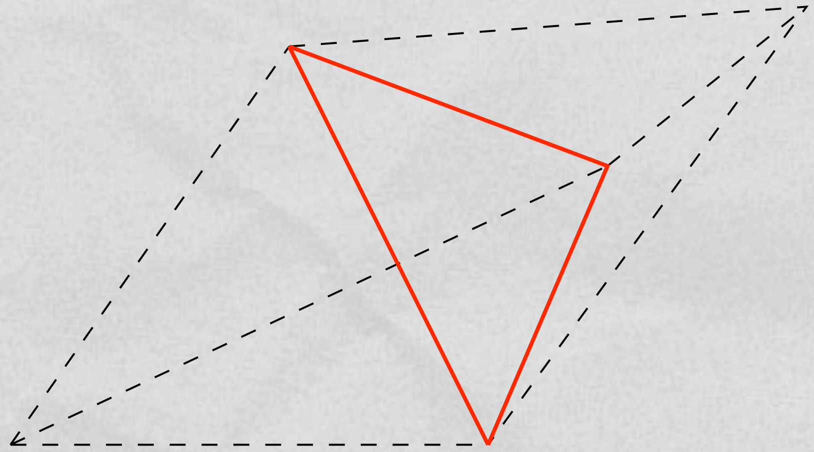
Final step in pictorial guide

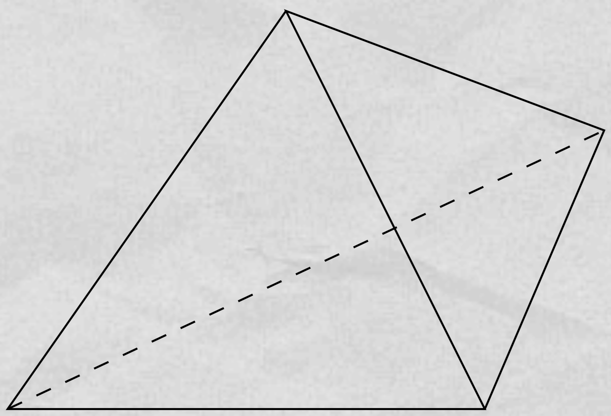
Field $\Phi(g_1, g_2, g_3)$



Action

$$\mathcal{S}[\phi, \phi^\dagger] \quad = \quad \frac{1}{2}\sum|\phi|^2 \quad + \quad \frac{\lambda}{4!}\sum\phi^4 \times \{6j - symbol\}$$





3d Group field theory - Boulatov Model

Quantum Theory - Partition Function

$$\mathcal{Z} \equiv \sum_{\Gamma} \frac{\lambda^n}{sym[\Gamma]} \mathcal{Z}[\Gamma]$$

The Feynman graphs may be thought of as

- *closed 4-valent graphs*
- *spin foam complex*
- *the 2-skeleton dual to a closed 3d triangulation* Δ

$\mathcal{Z}[\Gamma]$ *- the Feynman amplitude for the graph* Γ

$$\mathcal{Z}[\Gamma] = \sum_{j_e} \prod_{e \in \Delta} d_{j_e} \prod_{T \in \Delta} \{6j - symbol\}_T \quad \equiv \quad Z_{PR}[\Delta]$$

3d Group field theory - Boulatov Model

Boundary State Observables

Gauge-invariant products of fields

$$\mathcal{O}[F] = \prod_{v \in F} \phi^{j_i}_{m_i}$$

Basis given by spin networks

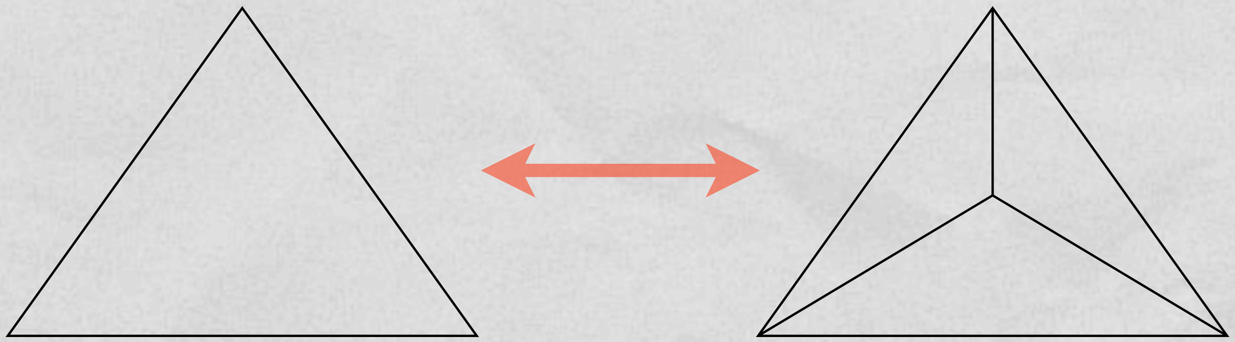
Represent boundary triangulation

3d Group field theory - Boulatov Model

Classical Theory - Equations of Motion

$$\Phi(g_1, g_2, g_3) = -\frac{\lambda}{3!}\int_{\mathcal{G}^3} \Phi(g_4, g_5, g_3)\Phi(g_4, g_2, g_6)\Phi(g_1, g_5, g_6)$$

This is the (1-3) Pachner move for 2d triangulations



Are there "instanton" solutions?

MOTIVATIONS

Emulate Matrix Models

Promotes the idea of higher tensor models

In the sum over representations QM ⟶ *QFT*

$$\mathcal{S}[\Phi,\Phi^\dagger] \quad = \quad \frac{1}{2}\int_{\mathcal{G}^2} |\Phi|^2(g_1,g_2) \quad + \quad \frac{\lambda}{3!}\int_{\mathcal{G}^3} \Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1)$$

↓ *Integrate out right shift symmetry*

$$\mathcal{S}[\varphi,\varphi^\dagger] \quad = \quad \frac{1}{2}\int_{\mathcal{G}^2} \varphi(g)\varphi(g^{-1}) \quad + \quad \frac{\lambda}{3!}\int_{\mathcal{G}^3} \varphi(g_a)\varphi(g_b)\varphi(g_c)\;\delta(g_a g_b g_c)$$

↓ *Transfer to momentum space*

$$\mathcal{S}[\varphi,\varphi^\dagger] \quad = \quad \sum_j d_j \left[\frac{1}{2}|\varphi^j|^2 \quad + \quad \frac{\lambda}{3!}\left[\varphi^j\right]^3\right]$$

Re-instate Triangulation Independence

4d-amplitudes are triangulation dependent
- *Barrett-Crane* — Barrett, Crane
- *New spin foams* — Engle, Pereira, Rovelli, Speziale, Livine, Freidel, Krasnov

Dealing with one spin foam truncates the degrees of freedom

Extended Boulatov model to 4d — Ooguri

Allowed for Barrett-Crane to be written down in GFT form — De Pietri, Freidel, Krasnov, Rovelli

$$\mathcal{S}[\phi,\phi^\dagger] \quad = \quad \frac{1}{2}\int_{\mathcal{G}^4} |P_g \phi|^2 + \int_{\mathcal{G}^{10}} [P_g \phi]^5 \rightarrow \text{4d BF theory}$$

$$\mathcal{S}[\phi,\phi^\dagger] \quad = \quad \frac{1}{2}\int_{\mathcal{G}^4} |P_g P_h \phi|^2 + \int_{\mathcal{G}^{10}} [P_g P_h \phi]^5$$

Representations of $\mathcal{S}^3 = SO(4)/SO(3)$ *are the balanced representations of SO(4)* — Freidel, Krasnov, Puzio

The new vertices has been given a GFT description — Freidel, Krasnov

Obtaining the Hamiltonian Constraint

Classical Equation of motion
akin to the
action of the hamiltonian constraint

Graviton propagator

One wants to take the continuum limit by a sum over spin foams

Proposal for physical inner product

Mini-superspace models

CONCRETE RESULTS

Any local spin foam amplitude can be realised as the Feynman amplitude of a GFT

$$Z[\Gamma] = \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_{f_e}, i_e) \prod_v A_v(j_{f_v}, i_{e_v})$$

Reisenberger-Rovelli

Observables in GFT

Perez-Rovelli

n-point functions of QFT → *n-net functions of GFT*

n-net functions can be defined in LQG

- *scalar under diffeomorphism*
- *fully gauge invariant*

n-net functions can be computed using the GFT

- *in general, formally*
- *but for free GFT, explicitly*

Provide a link between canonical and covariant approaches

Bubble Divergences

Perez-Rovelli

Divergences in spin foams are associated to bubbles rather than loops

$$\mathcal{S}[\phi,\phi^\dagger] = \frac{1}{2}\int_{SO(4)^4} |[P_g P_h \phi]|^2 \quad + \quad \frac{\lambda}{5!}\int_{SO(4)^{10}} [P_g P_h \phi]^5$$

Highlights the importance of the kinetic term in the action for convergence properties

Responsible for the finiteness of the amplitude

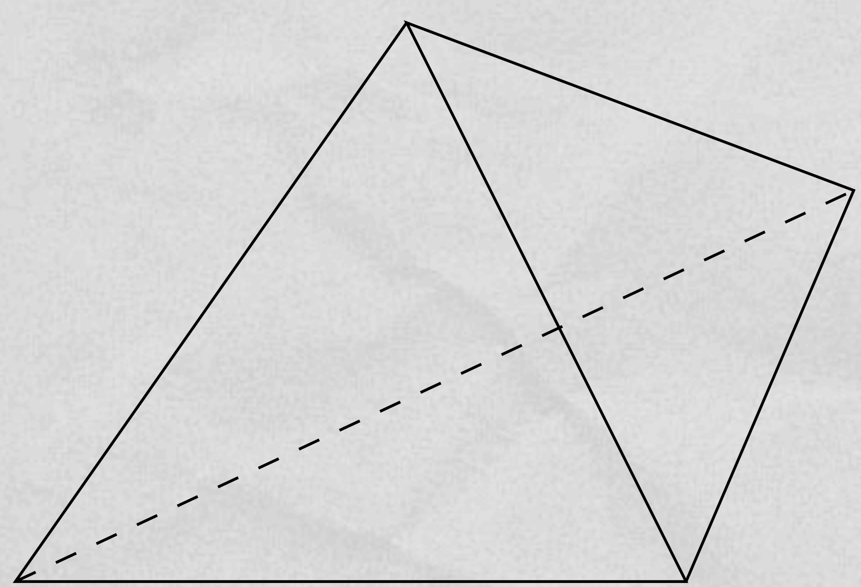
Argued not to be related to a quantisation of GR

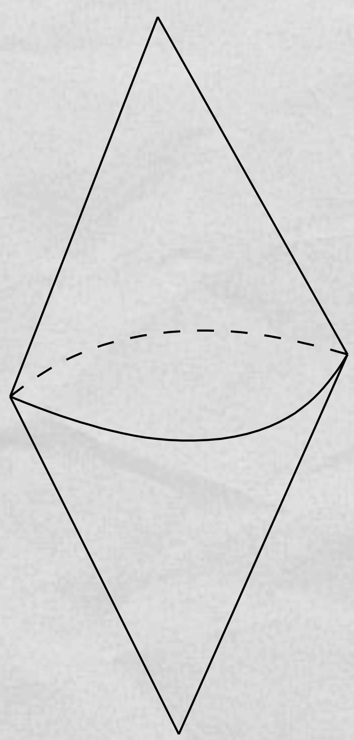
Resummability

Modification of the Boulatov model which is Borel resummable Freidel, Louapre

$$\boldsymbol{S}[\phi,\phi^{\dagger}] \quad = \quad \frac{1}{2}\int_{\mathcal{G}^3} \|\phi\|(g_1,g_2,g_3) \quad + \quad \frac{\lambda}{4!}\int_{\mathcal{G}^6}\Big[\,\delta\;\phi(g_1,g_2,g_3)\phi(g_4,g_5,g_3)\phi(g_4,g_2,g_6)\phi(g_1,g_5,g_6)$$

$$+\phi(g_1,g_2,g_3)\phi(g_4,g_5,g_3)\phi(g_5,g_4,g_6)\phi(g_2,g_1,g_6)\Big]$$





Matter in Group Field Theory

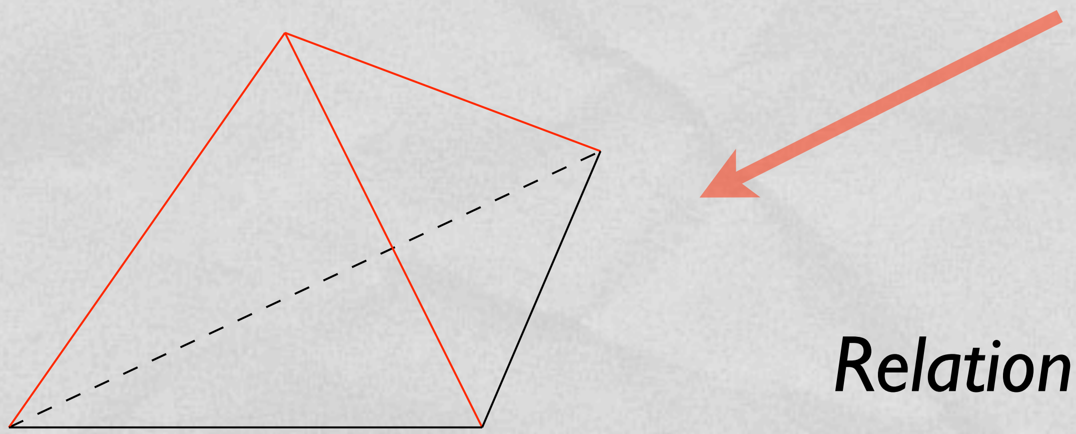
Impetus from spin foam model/loop quantum gravity — Freidel, Louapre, Livine, Noui, Perez

$$Z[\Gamma,\gamma] = \sum_{j_e} \prod_{e\notin\gamma} d_{j_e} \prod_{e\in\gamma} \chi^{j_e}(h_{\theta_e}) \prod_T \{6j\}_T \times \text{Spin amplitude}$$

GFT which generates these amplitudes — Freidel, Oriti, JR

$$\mathcal{S}[\phi,\phi^\dagger,\psi,\psi^\dagger] \quad = \quad \mathcal{S}_{Boulatov}[\phi,\phi^\dagger] \quad + \quad \frac{1}{2}\int |\psi_s|$$

$$+ \quad \frac{\mu}{3!}\int \psi_s\psi_s\psi_s\phi \quad \times \quad \text{Amplitude for tri-valent particle interaction}$$



Just like for geometry, all dynamics occurs in the vertex term

Relation to DSU(2) structure — Krasnov

Effective field theory for matter from GFT

EFT is a GFT — Freidel, Livine, Noui

$$\mathcal{S}[\varphi] = \frac{1}{2}\int_{SU(2)} \varphi(g)\underbrace{\left(P^2(g) - m^2\right)}\varphi(g^{-1}) + \int_{SU(2)^3} \varphi(g_1)\varphi(g_2)\varphi(g_3)\delta(g_1 g_2 g_3)$$

Non-trivial propagator

2d perturbation of 3d GFT — Fairbairn, Livine

Classical solution of Boulatov model

$$\phi_f(g_1, g_2, g_3) = \sqrt{\frac{3!}{\lambda}} \int_{\mathcal{G}} dg \quad \delta(g_1 g)\ f(g_2 g)\ \delta(g_3 g)$$

Substitute $\phi(g_1, g_2, g_3) = \phi_f(g_1, g_2, g_3) + \varphi(g_1, g_3)$

into the Boulatov action and examine first order perturbations around the classical solution → *For an appropriate choice of f, the result follows*

MODERN PERSPECTIVES

GFT as a common framework for simplicial quantum gravity

LQG: boundary states are spin network states

GFT field as 2nd quantised spin network functional Livine, Oriti

Spin foams

Dynamical Triangulations: Freeze field theory degrees of freedom

suggests causality conditions to tame some over triangulations

Hamiltonian Analysis of Generalised GFTs

Oriti, Ryan

Lagrangian/Hamiltonian

Introduce time variables in to the field

Introduce a propagator with a more interesting pole structure

$$\mathcal{S} = \frac{1}{2}\int_{(\mathcal{G}\times\mathbb{R})^4} \phi^\dagger \quad \mathcal{D} \quad \phi \quad + \quad \frac{\lambda}{5!}\int_{(\mathcal{G}\times\mathbb{R})^{10}} \phi^5$$

Proceed with the identification of the phase space, canonical analysis

Fock space of states, S-matrix

Emergence of the Continuum

Oriti

Indications are that the continuum limit will consist of a lot of simplices

a continuum space is a very large number of very small GFT quanta very close to equilibrium

Condensed Matter Viewpoint

Quantum gravity is about identifying the microscopic constituents of space and provide a tentative description of their microscopic dynamics

Transform from Hamiltonian picture to Statistical GFT

- a notion of temperature, equilibrium, vacuum state
- a Bose-Einstein condensate of particles

New Group Field Theories

Oriti, Tlas

Attempt to describe gravity in the first order formalism

Hold some information on causality

Link with Regge Calculus and dynamical triangulations

Classical Theory of GFTs

Freidel

Proposal for Physical Scalar Product

- *truncate the GFT expansion at tree level*

$$\langle F_1|F_2\rangle_0 = \sum_{\Gamma\in\mathcal{T}} \frac{\mathcal{Z}[\Gamma]}{sym[\Gamma]}$$

- *triangulation independent*
- *finite, since at tree level*
- *positive, but not strictly positive (it has a kernel)*

Loop expansion

$$\mathcal{S} = \frac{1}{2}\int |\phi|^2 \quad + \quad \frac{\lambda}{5!}\int \phi^5$$

$$\phi \to \frac{1}{\lambda^{\frac{1}{3}}}\phi \qquad\qquad \mathcal{S} = \frac{1}{\lambda^{\frac{2}{3}}}\left[\frac{1}{2}\int |\phi|^2 \quad + \quad \frac{1}{5!}\int \phi^5\right]$$

Classical Theory of GFTs

Loop expansion continued

$$\langle F_1|F_2\rangle_\alpha = \alpha^{-2}\sum_i \alpha^{2i}\langle F_1|F_2\rangle_i \qquad \alpha = \lambda^{\frac{1}{3}}$$

Schwinger-Dyson Equation

Search for Diffeomorphism Invariance

Ist Possibility: Schwinger-Dyson Equations of GFTs

- Quantum e.o.m. coming from an infinitesimal symmetry: Ward identity

2nd Possibility: In analogy with matrix models Livine, Ryan

- Quantum e.o.m. may be written as differential operators acting on the coupling constants

- These differential operators form an algebra which is related to the Virasoro algebra and so in turn to the diffeomorphism algebra of the circle

- Employed a similar procedure to higher tensor models in 3d

- Preliminary results: map to subalgebra of diffeomorphisms of a torus

CONCLUSION / OUTLOOK

Hamiltonian analysis is on the horizon

- *Fock space structure*
- *greater understanding of the kinetic term*
- *emergence of the continuum*

Analysis of Classical Theory

- *Proposal for physical inner product*
- *instanton solutions*

Matter as phase of 4d GFT

New GFT models with causal information

- *Relation to Regge calculus*
- *semi-classical limit*

Ward Identites and Schwinger-Dyson Equations

- *isolation of diffeomorphism invariance*

GENERAL DESCRIPTION

- Field and Symmetries: $\phi(g_1, \ldots, g_n)$

Field $\phi : \mathcal{G}^n \to \mathbb{C}$

Right shift symmetry

$$\phi(g_1 \bar{g}, \ldots, g_n \bar{g}) = \phi(g_1, \ldots, g_n) \qquad \text{for all } \bar{g} \in \mathcal{G}$$

Permutation symmetry

$$\phi(g_1, \ldots, g_n) = \phi(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \qquad \text{for all } \sigma \in S_n$$

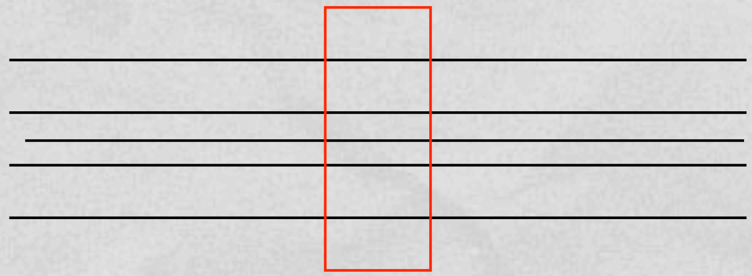
$$\phi(g_1, \ldots, g_n) = \phi^\dagger(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \qquad \text{for all } \sigma \in A_n$$

GENERAL DESCRIPTION

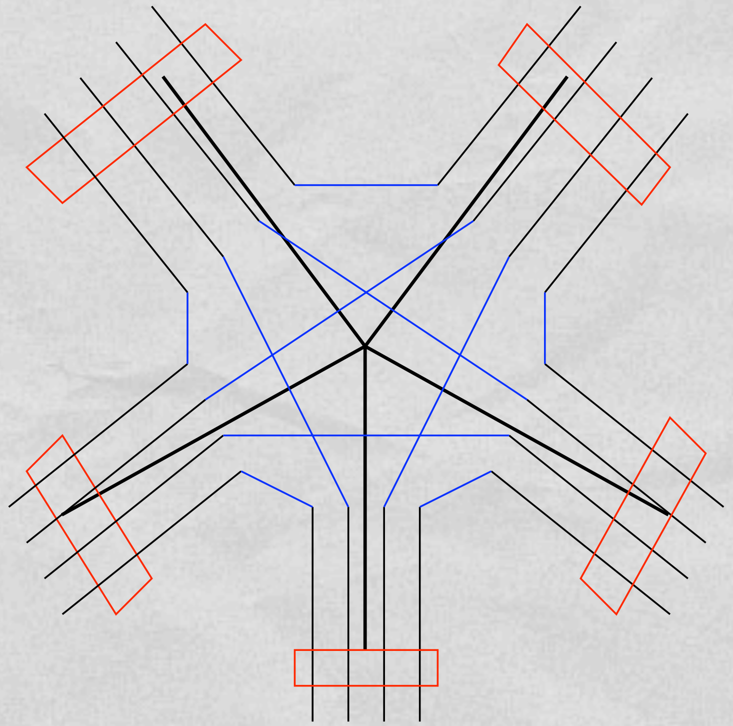
- Action:

$$\mathcal{S}[\phi,\phi^\dagger] = \frac{1}{2}\int_{\mathcal{G}^4} \phi^\dagger\phi(g_i) + \frac{\lambda}{(n+1)!}\int_{\mathcal{G}^{n(n-1)}} \phi^5(g_i)\ \phi^{n+1}$$

Kinematic Term



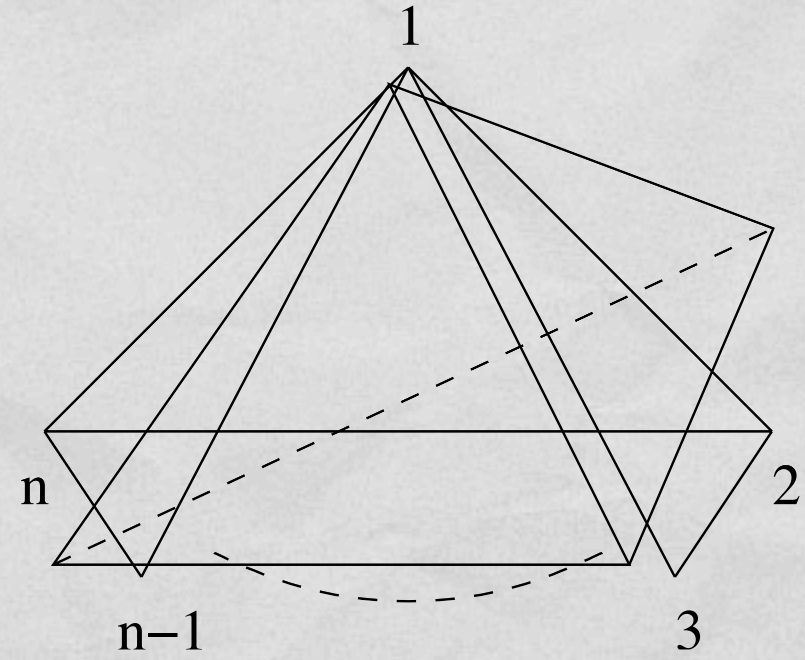
Vertex Term



GENERAL DESCRIPTION

Field and Symmetries: $\phi(g_1, g_2, g_3, g_4)$ / $\phi(g_1, g_2, ..., g_n)$

Field represents a fundamental
3-simplex / (n-1)-simplex



Right shift symmetry
ensures gauge symmetry

Permutation symmetry ensures that
the fundamental simplices are oriented

GENERAL DESCRIPTION

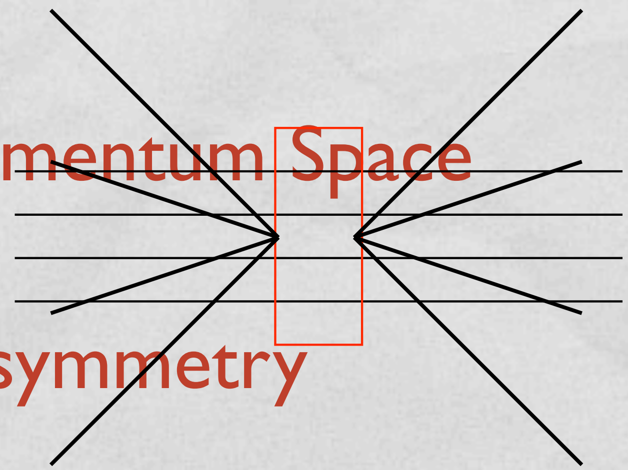
Field and Symmetries:

$\phi(g_1, \ldots, g_n)$

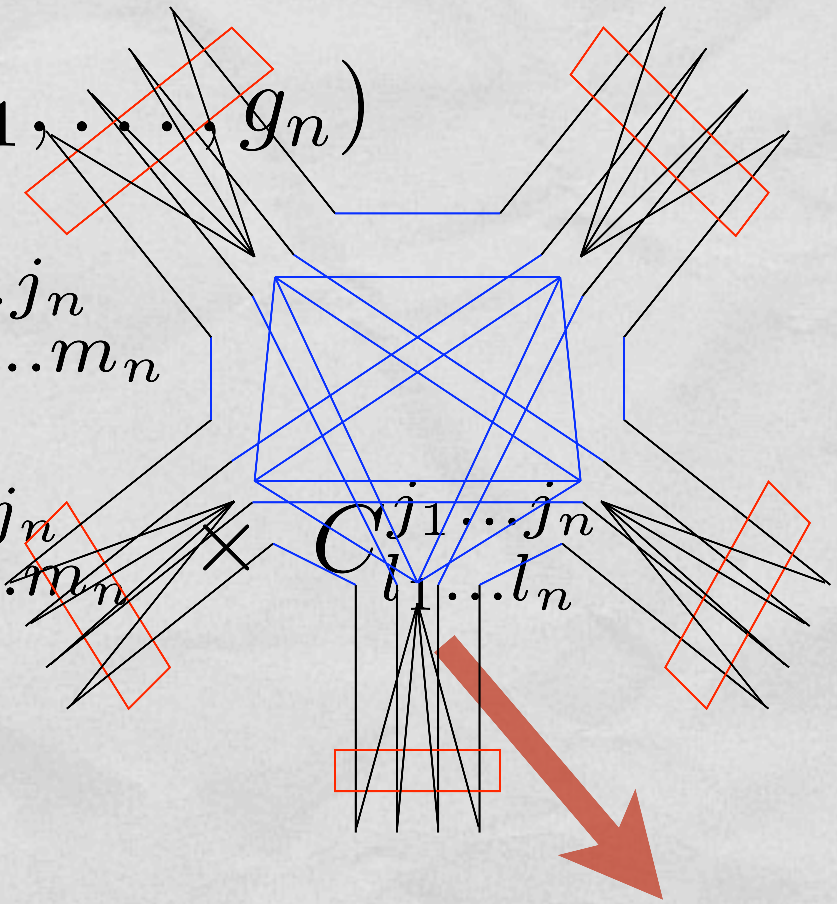
Field in Momentum Space

$\phi^{j_1 \ldots j_n}_{m_1 \ldots m_n}$

Right shift symmetry



$\phi^{j_1 \ldots j_n}_{m_1 \ldots m_n} \times C^{j_1 \ldots j_n}_{l_1 \ldots l_n}$

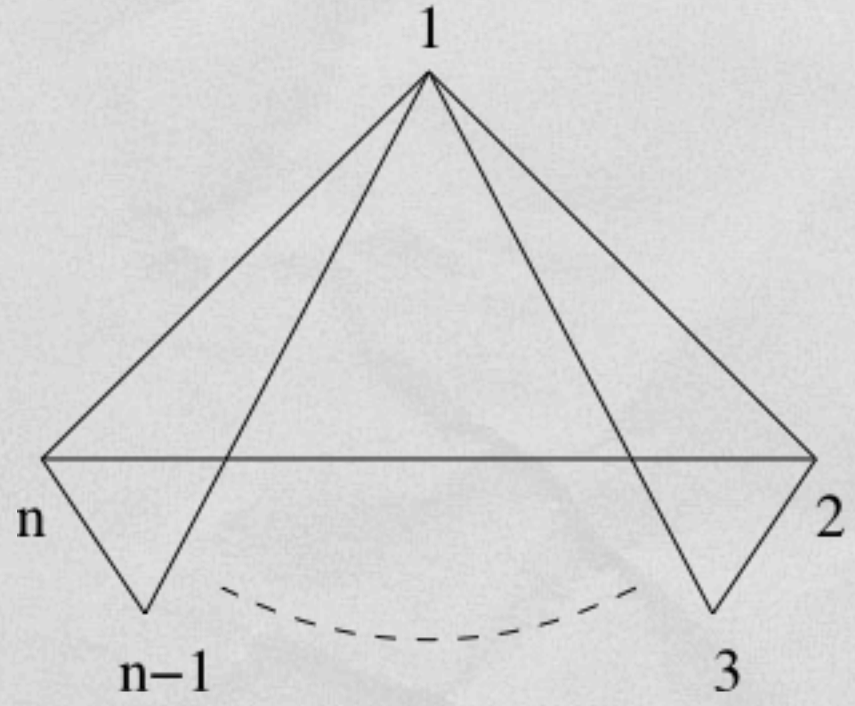


Action

$$S[\phi, \phi^\dagger] = \frac{1}{2} \sum \phi^{\dagger j_i}_{m_i} \phi^{j_i}_{m_i} + \frac{\lambda}{(n+1)!} \sum \phi^{n+1}_{j} \times \{symbol\}$$

GENERAL DESCRIPTION

- States: (*n-1*)-geometries



Simplicial superspace

Glued simplicial building blocks

↓

Reconstruct *topology*

Reconstruct *geometry*

GENERAL DESCRIPTION

Classical Equations of Motion

$$\phi = \frac{\lambda}{n!}\phi^n$$

- Familiar *1-n* Pachner move for an *(n-1)*-dimensional triangulation
- There exist solutions