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Spherically symmetric loop quantum gravity: towards the complete space-time

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Plan

- Introduction: previous work, rough outline.
- Spherically symmetric gravity
- Gauge fixing and quantization.
- A solution for the holonomized theory.
- Some initial findings for the solution.
- Future work.

Previous work:

Spherically symmetric space-times include Schwarzschild and therefore the singularity.

They are the “next obvious thing” to try with loop quantum gravity after homogeneous space-times.

Unlike the homogeneous case, even after imposing spherically adapted coordinates one ends up with an algebra of constraints with structure functions. So either one uses non-traditional methods like the “master constraint” or the “consistent discretizations”, or one extends further the partial gauge fixing to simplify the algebra.

This was done successfully for the exterior problem,
R. Gambini, M. Campiglia, JP, Class. Quan. Grav. 24, 3649 (2007).

and also for the interior problem
R. Gambini, M. Campiglia, JP, arxiv:0712.0817 (2007)
C. Boehmer, K. Vandersloot, Phys. Rev. D76, 104030 (2007).

Here we would like to discuss the application of similar techniques to the complete space-time with spherical symmetry.

Rough outline:

We take spherically symmetric general relativity written in terms of the new variables a la Bojowald-Swiderski. The model has one diffeomorphism constraint and a Hamiltonian constraint. The constraints have the usual algebra with structure functions.

We partially gauge fix and get rid of the diffeomorphism constraint. The Hamiltonian constraint becomes Abelian and the theory has a true Hamiltonian.

We holonomize the Hamiltonian constraint in the usual way.

We do not discuss the quantization in detail, but jump to study solutions of the holonomized classical theory, since one knows they capture the semiclassical behavior of the quantum theory.

To find the solution one has to further gauge fix. There is freedom in how to do it, and therefore in the solution found. The counterpart in the quantum theory of this further gauge fixing would be the introduction of a family of evolving constants of the motion.

We carry out a preliminary analysis of the classical solution found.

Spherical symmetry with the new variables

Previous work with the new variables, Bengtsson (1988) Kastrup and Thiemann (1993) and Bojowald and Swiderski (2005, 2006). Choose connections and triads adapted to spherical symmetry,

$$A = A_x(x)\Lambda_3 dr + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)\, d\theta + ((A_1(x)\Lambda_2 - A_2(x)\Lambda_1)\sin\theta + \Lambda_3\cos\theta)\, d\varphi,$$

$$E = E^x(x)\Lambda_3 \sin\theta \frac{\partial}{\partial x} + \left(E^1(x)\Lambda_1 + E^2(x)\Lambda_2\right)\sin\theta\frac{\partial}{\partial\theta} + \left(E^1(x)\Lambda_2 - E^2(x)\Lambda_1\right)\frac{\partial}{\partial\varphi},$$

Λ's are generators of su(2).

It simplifies the constraints if one introduces a “polar” canonical transformation in the variables A_φ, P^φ, β, P^β

$$\begin{aligned} A_1 &= A_\varphi \cos\beta, \\ A_2 &= -A_\varphi \sin\beta, \\ E^\varphi &= \sqrt{(E^1)^2 + (E^2)^2}. \end{aligned}$$

$$\begin{aligned} P^\varphi &= 2E^1\cos\beta - 2E^2\sin\beta, \\ P^\beta &= -2E^1 A_\varphi \sin\beta + 2E^2 A_\varphi \cos\beta, \end{aligned}$$

To fix asymptotic problems (Bojowald, Swiderski), one does a further canonical change,

$$\begin{aligned} A_\varphi \to \bar{A}_\varphi &= 2\cos\alpha A_\varphi, \\ \beta \to \eta &= \alpha + \beta, \end{aligned}$$

$$P^\beta = P^\eta, \quad P^\varphi = 2E^\varphi\cos\alpha,$$

Leading to the canonical pairs $A_x, E^x, \bar{A}_\varphi, E^\varphi, \eta, P^\eta$.

One can introduce gauge invariant variables (Gauss' law is then gone),

$$2\gamma K_x = A_x + \eta' \quad A_\varphi = 2\gamma K_\varphi.$$

The canonically conjugate pairs are now E^x, K_x and E^φ, K_φ.

The relation to the traditional metric variables is,

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\mathrm{sign}(E^x)\frac{(E^\varphi)^2}{\sqrt{|E^x|}}(A_x + \eta'), \qquad K_{\theta\theta} = -\sqrt{|E^x|}\frac{A_\varphi}{2\gamma},$$

The constraints take a relatively simple form with the usual 1+1 diffeo/Hamiltonian algebra of constraints (with structure functions),

$$D = -(E_x)' K_x + E^\varphi (K_\varphi)'$$

$$H = -\frac{1}{2}\frac{E^\varphi}{\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{1}{2}\frac{E^\varphi K_\varphi^2}{\sqrt{E^x}\gamma} + \frac{1}{8}\frac{\left((E^x)'\right)^2}{\sqrt{E^x}E^\varphi} - \frac{1}{2}\frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{(E^\varphi)^2} + \frac{1}{2}\frac{\sqrt{E^x}(E^x)''}{E^\varphi},$$

The quantization of this model directly is therefore a hard thing since it has the "problem of dynamics". One could use the master constraint or the consistent discretizations.

We start by fixing a gauge $E^x = f(x,t)$. This determines the Lagrange multiplier (shift) $N^r = \dot{f}(x,t)/f'(x,t)$ The variable K_x is eliminated imposing the diffeomorphism constraint strongly.

One is left with E^φ and K_φ as canonical variables and with one (Abelian) constraint and a true Hamiltonian, since the gauge fixing breaks reparametrization invariance,

$$H_{\text{true}} = \int dx \frac{\dot{f}(x,t)}{f'(x,t)} E^\varphi \left(K_\varphi\right)'$$

$$H_T = \int dx N' \Phi + H_{\text{True}} + H_{\text{Boundary}},$$

$$\Phi = -\sqrt{E^x} - K_\varphi^2 \sqrt{E^x} + \frac{1}{4} \frac{\left((E^x)'\right)^2 \sqrt{E^x}}{\left(E^\varphi\right)^2} + 2M$$

For reasons of time we omit discussion of the boundaries, where things proceed essentially in the same way as in the exterior case.

Loop representation for the spherically symmetric case:

Manifold is a line. “Graph” is a set of edges $g = \bigcup_i e_i$. The only variable that behaves as a connnection on the line is A_x. The variables η and A_φ are scalars, so in the loop representation one uses “point holonomies” to represent them.

To avoid presenting too many equations, I will write the states for the “gauge fixed” case we introduced. There the only variables in the bulk are E^φ and $2\gamma K_\varphi = A_\varphi$

$$\mathcal{H} = L^2\left(\otimes_N R_{\rm Bohr}, \otimes_N d\mu_0\right)$$

$$T_{g,\vec{\mu}}[K_\varphi] = \prod_{v\in V(g)} \exp\left(2i\mu_v \gamma K_\varphi(v)\right)$$

$$\hat{E}^\varphi_m = -i\ell^2_{\rm Planck}\frac{\partial}{\partial K_{\varphi,m}},$$

$$\hat{E}^\varphi_m T_{g,\vec{\mu}} = \sum_{v\in V(g)} \mu_m \gamma \ell^2_{\rm Planck}\delta_{m,n(v)} T_{g,\vec{\mu}},$$

Volume of an interval I

$$V(I) = 4\pi \sum_{m\in I} |E^\varphi_m| \sqrt{|E^x|}$$

c-number

$$\hat{V}(I) T_{g,\vec{\rho}} = \sum_{v\in I} 4\pi |\rho_v| \sqrt{|E^x|}\ell^2_{\rm Planck} T_{g,\vec{\rho}}.$$

Study of the classical holonomized theory:

Starting from the total Hamiltonian,

$$H_T = -\int dx N' \left[-\sqrt{E^x} - K_\varphi^2 \sqrt{E^x} + \frac{1}{4} \frac{\left((E^x)' \right)^2 \sqrt{E^x}}{(E^\varphi)^2} + 2M \right] + \int dx N^r E^\varphi \left(K_\varphi \right)',$$

We discretize it on a lattice as we described and holonomize

$$\begin{aligned} H_T^{\mu,\epsilon} \;=\; & -\sum_0^L \left(N_{n+1} - N_n \right) \left[-\sqrt{E_n^x} - \frac{\sin\left(\mu K_{\varphi,n}\right)^2}{\mu^2} \sqrt{E_n^x} + \frac{1}{4} \frac{\left(E_{n+1}^x - E_n^x \right)^2 \sqrt{E_n^x}}{\left(E_n^\varphi \right)^2} + 2M \right] \\ & + \sum_0^L N_n^r E_n^\varphi \frac{\exp\left(i\mu \left(K_{\varphi,n+1} - K_{\varphi,n} \right) \right) - 1}{i\mu}, \end{aligned}$$

To simplify, we will work in the continuum limit in which we make the separation of the points of the lattice vanish but keep μ finite.

$$H_T^\mu = -\int dx N' \left[-\sqrt{E^x} - \frac{\sin\left(\mu K_\varphi\right)^2}{\mu^2} \sqrt{E^x} + \frac{1}{4} \frac{\left((E^x)' \right)^2 \sqrt{E^x}}{(E^\varphi)^2} + 2M \right] + \int dx N^r E^\varphi \left(K_\varphi \right)'.$$

Recall that N^r is a given function, $N^r = \dot{f}(x,t)/f'(x,t)$

We will proceed to find a solution of the theory. That is, a solution of the constraint and of the evolution equations. A solution has to be found in a given gauge. So we will need to further gauge fix to have explicit results.

We would like to choose f(x,t) in such a way that in the limit μ->0 one recovers the usual Schwarzschild solution in Kruskal-like coordinates. That is, a metric with a singularity at $x^2-t^2=-1$. On the other hand, for finite μ, we would like that surface to be a regular surface beyond which we can extend the metric.

We will choose $E^x=f_1(u,t,\delta)$ and $K_\varphi=f_2(u,t,\delta)$ with $u=x^2-t^2+1$ and δ a positive parameter such that when δ->0, μ->0 we get the usual Schwarzschild/Kruskal solution.

We require the following of the gauge fixing: The variable u goes from 0 to infinity and is such that the radial variable depends logarithmically on it,

$$r = \sqrt{E^x} \sim \ln(u) \text{ for } u \to \infty.$$

One has that

$$E^\varphi \sim r + M \quad \sim \ln(u)/\sqrt{u} + const. + O(\ln(u)/u)$$

$$K_x \sim K_\varphi \sim 1/\sqrt{u}.$$

At u=0 we require all variables to be finite and t-independent and will choose their derivatives to vanish. That makes it easy to extend beyond u=0 without introducing shells of matter. There may be many other choices possible. Although it appears one has a lot of freedom, it is not easy in the end to make everything work out.

After quite a bit of trial and error, our (current) proposal for E^x is,

$$E^x = M^2 \left\{ \frac{\left[\delta\left(1+u\right) + b^2\left(10u^{3/2}+u^3\right)\left(\delta\left(t^2-1\right)+1\right)\right]}{u^3+\left(t^2-1\right)\left(\delta u^3+\delta^2\right)+\frac{1}{3}\delta^2 u} \ln(1+u) + \delta^2 \right\}$$

With b a function of t that we will later see varies slowly. This choice has, for u->0 E^x->$M^2\delta^2$ independent of t, for large u it goes as $b^2\ln(u)^2$+const.+O(ln(u)/u) and for δ->0 it goes to zero for u=0 giving rise to the singularity.

A technicality is that this choice is valid only for |t|>1. This can be extended easily, but the expressions become even lengthier.

For K_φ we chose,

$$K_\varphi = \frac{1}{2}\pi \frac{\left(1+\ln\left(1+u^2\right)\right)\delta^{5/2}}{\mu\left(\delta^{5/2}+\ln^2\left(1+u\right)\right)} + \frac{t\ln\left(1+u^4\right)}{u^{3/8}\left(1+u^{1/8}\right)\left(t\delta^8+\ln\left(1+u^4\right)\left(1+\delta t\right)\right)}.$$

This choice has, for u->0, that $K_\varphi=\pi/(2\mu)$ which implies $\sin(\mu K_\varphi)\sim 1$ and the holonomized theory has maximal departure from usual GR at the place where one would have expected the singularity. In the limit δ->0 it blows up at u=0 and one has the usual singularity. Its first derivative with respect to x vanishes at u=0.

We then solve the constraint for E^{φ},

$$E^{\varphi} = \frac{1}{2}(E^x)' \left(\sqrt{1 - \frac{2M}{\sqrt{E^x}} + \frac{\sin(\mu K_{\varphi})^2}{\mu^2}}\right)^{-1}$$

Since the numerator vanishes for u=0 and one wishes E^{φ} to be finite there, this leads to a relation between δ and μ.

$$\mu = \frac{\delta^2}{\sqrt{2-\delta^4}}.$$

The derivative vanishes at u=0 as we wanted and the large u behavior implies that

$$g_{xx}|_{u\to\infty} = 4\frac{M^2b^2}{u} + 8\frac{M^2 b}{u\ln(u)},$$

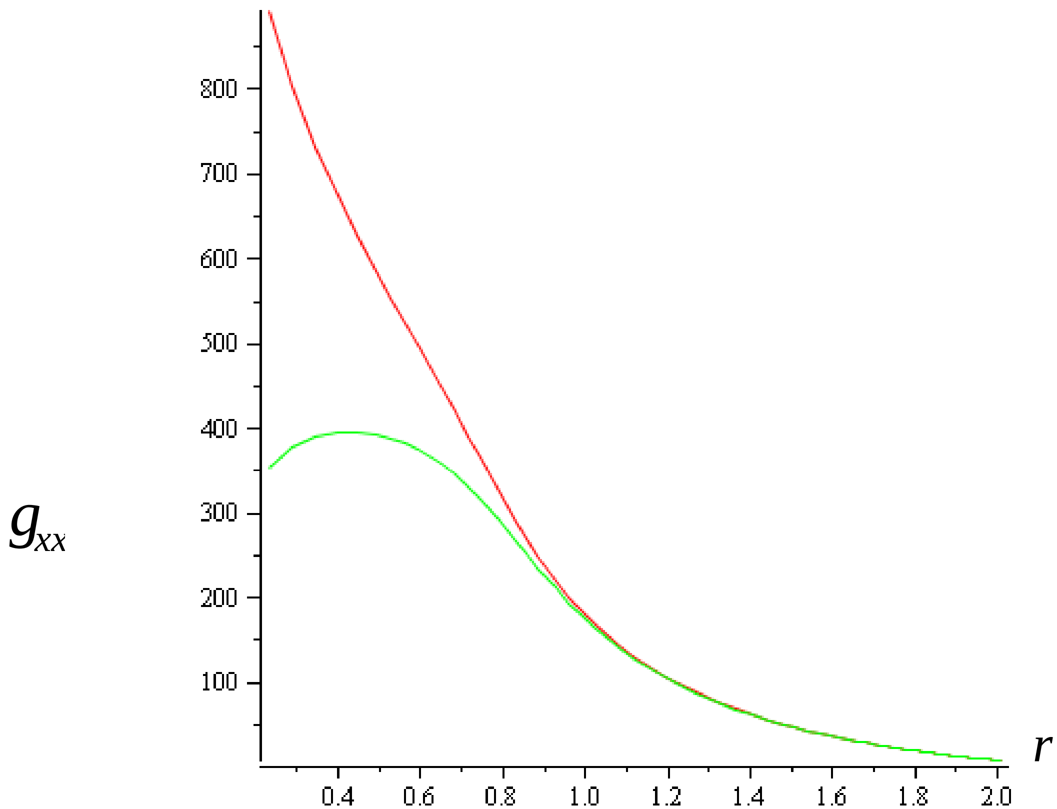
Which is just (1+2m/r) in these coordinates.

The evolution equations determine the lapse up to a quadrature, that can be done numerically.

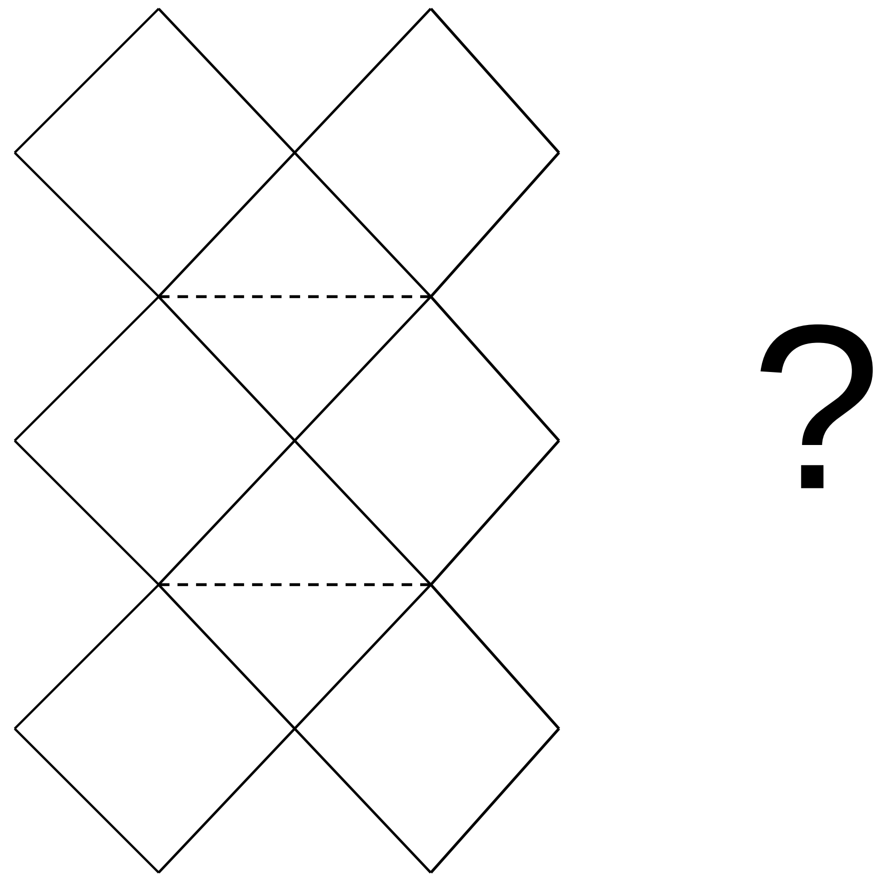
$$N' = \frac{1}{4}\frac{\dot{K}_{\varphi}(E^x)' - K'_{\varphi}\dot{E}^x}{\left(1 - \frac{2M}{\sqrt{E^x}}\right)^{3/2}\sqrt{E^x}}$$

This completes the determination of the solution, which we only have started to analyze. It is harder than it looks since the expressions are large and some are only known numerically.

To get a feel for what is going on, it is useful to plot g_{xx} as a function of r. The red curve is the Kruskal-like solution we get in the limit $\delta \to 0$, the green curve the solution of the holonomized theory. This is for $\delta=10^{-6}$ and for t=1.5.



Global picture?



Assuming things work out at i^+, i^-…

Summary:

- One can study spherically symmetric space-times using loop quantum gravity.
- One needs to use special features of spherical symmetry to apply the traditional Dirac quantization technique.
- The holonomized theory can be solved, but the solution has a somewhat complicated form, which requires a more delicate (numerical) analysis to determine its global structure.