Self-adjointness of Hamiltonian constraint in LQC

WORKSHOP “LOOPS & FOAMS”

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L. Szulc, W. Kamiński, J. Lewandowski: arXiv:gr-qc/0612101 $(k = 1)$

W. Kamiński, J. Lewandowski: arXiv:0709.3120 $(k = 0, \Lambda \leq 0)$

L. Szulc: arXiv:0707.1816 $(k = -1, \Lambda < 0)$

W. Kamiński, TP: *work in progress* $(k = 0, \Lambda \geq 0)$

A. Ashtekar, E Bentivegna, TP *in prep* $(k = 0, \Lambda > 0)$

The model

Flat/spherical isotropic universe (FRW: $k = 0, 1, -1$) with massless scalar field + cosmological constant (Abhays's talk).

- Quantization method:
  - Matter DOF: standard (Schrodinger)

    Kinematic Hilbert space: $\mathcal{H}^{\text{kin}}_{\phi} = L^2(\mathbb{R}, \mathrm{d}\phi)$,

    Configuration variable: ϕ
  - Geometry DOF: methods of LQG

    Kinematic Hilbert space: $L^2(\bar{\mathbb{R}}_{\text{Bohr}}, \mathrm{d}\mu_{\text{Bohr}})$,

    configuration variable: $v \propto$ **volume of fiducial cell/3d Cauchy surface**

    basis: $|v\rangle$, $\langle v|v'\rangle = \delta_{v,v'}$
- Kinematical Hilbert space: $\mathcal{H}^{\text{kin}} = \mathcal{H}^{\text{kin}}_{\text{grav}} \otimes \mathcal{H}^{\text{kin}}_{\phi}$
- Scalar constraint: $\partial^2_{\phi}\Psi(v,\phi) = -\Theta\Psi(v,\phi)$; Θ – difference operator
- Reinterpretation: free system evolving in ϕ; Θ – evolution operator
- Superselection: $\mathcal{H}^{\text{kin}}_{\text{grav}} = \bigotimes_{\varepsilon} \mathcal{H}^{\text{kin}}_{\text{grav},\epsilon}$ where $\mathcal{H}^{\text{kin}}_{\text{grav},\epsilon}$ – restriction of $\mathcal{H}^{\text{kin}}_{\text{grav}}$ to functions supported on sets $\mathcal{L}_{\varepsilon} = \{\varepsilon + 4n; n \in \mathbb{Z}\}$ preserved by Θ.

Question & Outline

- Question:
  - Does Θ unambiguously define dynamics ?
    Mathematically: Is it (essentially) self-adjoint ?

- Outline: *Considered model:* $k = 0$ *unless specified othervise.*
  - Form and general properties of Θ
  - $k = 0, \Lambda = 0$;
  - $k = 0, \Lambda < 0$; $(k = 1, \Lambda \leq 0; k = -1, \Lambda < 0)$
  - $k = 0, \Lambda > 0$;
  - Summary

Evolution operator

- $\Theta = \Theta_o - \Lambda V_\Lambda(v)\mathbb{I} + \delta_{k,1} V_S(v)\mathbb{I} - \delta_{k,-1} V_H(v)\mathbb{I}$

  $\Theta_o = B(v)^{-\frac{1}{2}}[-\hat{h}_2 A(v)\hat{h}_2 - \hat{h}_{-2} A(v)\hat{h}_{-2} + A(v+2) + A(v-2)]B(v)^{-\frac{1}{2}}$

  $\hat{h}_\lambda|v\rangle = |v+\lambda\rangle$

  $A := 3\pi G/8\,|v| \cdot ||v+1| - |v-1||$

  $B := 27/8\,|v|\,\left||v+1|^{1/3} - |v-1|^{1/3}\right|^3$

  $V_\Lambda := \beta|v|B(v)^{-1}$, $\beta = \mathrm{const}$; for large $|v|$: $V_\Lambda \propto |v|^2$

  V_S - (complicated) positively definite; for large $|v|$: $V_S \propto |v|^{4/3}$

  $V_H = \tilde{\beta}|v|^{1/3}||v+1| - |v-1||$.

- Properties:
  - Explicitely symmetric on $\mathcal{D}_\varepsilon$ – set of finite linear combinations of $|v\rangle$, where $v \in \mathcal{L}_\epsilon$.
  - Positively definite for $\Lambda \leq 0$ in cases $k = 0, 1$.

$\Lambda = 0$

- The results:
  - Θ_o is essentially self-adjoint.
  - Continuous part of its spectrum: $[0, \infty)$.
  - Essential spectrum: $[0, \infty)$.
  - Discrete and singular parts empty.
- Method of proof:
  - Thm 1: A - essentially self-adjoint, B - symm. and of trace class, $\Rightarrow$
    - $A + B$ - essentially self-adjoint
    - continuous and essantial parts of $\mathrm{Sp}(A)$ and $\mathrm{Sp}(A + B)$ are (respectively) the same.
  - We show that: Θ_o is up to trace class perturbation some operator of well known properties.

$\Lambda = 0$: *the sketch of proof (1)*

- Def: Operator A is of the trace class $\Leftrightarrow \mathrm{Tr}(A) < \infty$
- Perturbation:
  - $$\Theta_o = -\hat{h}_2 F(v) \hat{h}_2 - \hat{h}_{-2} F(v) \hat{h}_{-2} + G(v)\mathbb{I}$$
    $F = A(v)[B(v+2)]^{-1/2}[B(v-2)]^{-1/2} = \frac{3\pi G}{4}(v^2 - 2 - \alpha) + O(v^{-2})$
    $G = [A(v+2) + A(v-2)][B(v)]^{-1} = \frac{3\pi G}{2}(v^2 - \alpha) + O(v^{-2})$
    $\alpha = 5/9$
  - Take the dominant parts only:
    $$\Theta_o' := \hat{h}_2 F'(v) \hat{h}_2 + \hat{h}_{-2} F'(v) \hat{h}_{-2} + G'(v)\mathbb{I}$$
  - The difference:
    $$\Theta_o - \Theta_o' := \hat{h}_2 F'(v) \hat{h}_2 + \hat{h}_{-2} F'(v) \hat{h}_{-2} + G'(v)\mathbb{I}$$
    is a trace class operator.

$\Lambda = 0$: *the sketch of proof (2)*

- Set of unitary transformations:
  - Fourier transform: $\mathcal{F} : \mathcal{H}^{\mathrm{kin}}_{\mathrm{grav},\epsilon} \to L^2([0,2\pi])$

    $$\psi \in \mathcal{D}_\varepsilon \to (\mathcal{F}\psi)(x) = \sum_{n\in\mathbb{Z}} \psi(\varepsilon + 4n) e^{i(n+\varepsilon/4)x}$$

    Transformed operator: $\Theta'' = \mathcal{F}\Theta'\mathcal{F}^{-1}$

    $$\Theta'' = -3\pi G \left[\left(4\sin\left(\tfrac{x}{2}\right)\partial_x + \cos\left(\tfrac{x}{2}\right)\right)^2 + (\alpha+1)\sin^2\left(\tfrac{x}{2}\right)\right]$$

    Domain: $\mathcal{D}(\Theta'') := C^\infty_{0,2\pi}([0,2\pi]) =$

    $\{f \in C^\infty([0,2\pi]);\ f = 0$ on the neigh. of $0, 2\pi\}$
  - Variable transform: $y(x) = \frac{1}{2}\ln\tan\left(\frac{x}{4}\right)$

    Transformed operator:

    $$\Theta''' = W^{-1}\Theta'' W = 12\pi G\left(-\partial_y^2 - \frac{\alpha+1}{4\cosh^2(\alpha y)}\right)$$

    Domain: $C^\infty_0(\mathbb{R})$
- Properties of Θ'''
  - is essentially self-adjoint,
  - absolutely cont. part of spectrum is $[0,\infty)$,
  - abs. cont. part is unitarily equivalent to $-12\pi G\partial_y^2$.

$\Lambda < 0$

- The results:
  - Θ is essentially self-adjoint,
  - its spectrum is discrete,
  - density of eigenvalues $\lambda < E$ is bounded from above by the number of elements of: $\{n \in \mathbb{Z};\ \beta|\Lambda||v|[B(v)]^{-1} < E\}$
- Method of proof:
  - Thm 2: Suppose C_1 is essentially self-adjoint, its spectrum is discrete.
    Then the operator $C = C_1 + C_2$, where
    $$\|C_1\psi\|^2 \leq \|C_2\psi\|^2 + \beta' \|\psi\|^2$$
    (for some $\beta' > 0$) is also essentially self-adjoint and its spectrum is discrete.
  - We show that: Above inequality is satisfied for $C = \Theta$ and C_1 being some simple diagonal operator.

$\Lambda < 0$: *the sketch of proof* (1)

- The inequality:
  - $\Theta = \Theta_1 + \Theta_2$
    $\Theta_1 = -\hat{h}_2 F(v)\hat{h}_2 - \hat{h}_{-2}F(v)\hat{h}_{-2}, \qquad \Theta_2 = G(v)\mathbb{I},$
    $F = A(v)[B(v+2)]^{-1/2}[B(v-2)]^{-1/2} = \frac{3\pi G}{4}(v^2-2-\alpha)+O(v^{-2}),$
    $G = [A(v+2)+A(v-2)-\beta\Lambda|v|]\frac{1}{B(v)} = (\frac{3\pi G}{2}-\beta\Lambda)(v^2-\alpha)+O(v^{-2}).$
  - $\|\Theta_1\psi\|^2 \leq 4\|G\psi\|^2 \leq 4(\|F\psi\|^2 + 2\|(FF)^{1/2}\psi\|^2 + \|F\psi\|^2) = (\star)$
  - Given a bounded f the operator $f(v)\mathbb{I}$ is bounded on $\mathcal{D}_\varepsilon$, thus $\exists\beta', \tilde{\beta}' > 0$ s.t. $(\star) \leq \|G\psi\|^2 + \tilde{\beta}'\|\psi\|^2 \leq \|\Theta_2\psi\|^2 + \beta'\|\psi\|^2$
- Bound on number of eigenfunctions:
  - Given an operator $-\beta\Lambda|v|/B(v)\mathbb{I}$ all states $|\varepsilon+4n\rangle$, $n\in\mathbb{Z}$ are eigenstates.
    Number of $\lambda < E$ is a number of n s.t. $\beta|\Lambda||\varepsilon+4n|/B(\varepsilon+4n) < E$.
  - $\|\Theta\| \geq \|-\beta\Lambda|v|[B(v)]^{-1}\mathbb{I}\|$.

$\Lambda < 0$: *the sketch of proof* (2)

- The extensions:
  - It is enough that inequality of Thm.2 is satisfied outside of compact set of v. Thus since for large $|v|$ V_Λ dominates over V_S the results hold also for $k = \pm 1$ (with possibly different estimates on number of $\lambda < E$).
  - Analogous estimates can be made for $\Lambda = 0$, $k = 1$ thus the results hold also in that case.

$\Lambda > 0$

- The results:
  - Θ is NOT essentially self-adjoint. There exists many extensions.
  - Family of extensions is 4-dimensional, and
  - spectrum of each extension of Θ is discrete.
- Method of proof:
  - Thm 3: Suppose that for given Θ the spaces $\mathcal{K}_\pm$ of normalizable solutions to $\Theta\phi_\pm = \pm i\phi_\pm$ (the deficiency subspaces) are nontrivial and have equal dimension. Then:
    - Θ has many self-adjoint extensions,
    - there is $1-1$ correspondence between set of extensions and set of unitary operators $U_\alpha : \mathcal{K}_+ \to \mathcal{K}_-$
  - We show that: all (elements of 2-dimensional space of) deficiency functions are normalizable.

$\Lambda > 0$: *the sketch of proof* (1)

- The decomposition matrix
  - The (difference) equation $\Theta|\psi\rangle = \lambda|\psi\rangle$, $\lambda \in \mathbb{C}$ is a 2nd order one and can be written in matrix form
    $$\tilde{\psi}_n = M_{\Theta,n}\tilde{\psi}_{n-1}\ , \qquad \tilde{\psi}_n = (\psi_{n+1}, \psi_n)^T$$
    where M_Θ – known 2×2 matrix and $\psi_n = \langle \varepsilon + 4n|\psi\rangle$.
  - $\tilde{\psi}_n$ can be writen as linear combination of functions
    $$e_\pm(v) = |v|^{-1}e^{i\omega v},$$
    where $\omega(\Lambda)$ satisfies $\cos(4\omega) = 1 - (2\beta/3\pi G)\Lambda$.
    $$\tilde{\psi}_n = \begin{pmatrix} e_+(\varepsilon + 4(n+1)) & e_-(\varepsilon + 4(n+1)) \\ e_+(\varepsilon + 4n) & e_-(\varepsilon + 4n) \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = A_n\alpha_n$$
  - Equation can be written as $\alpha_n = B_n\alpha_{n-1}$ where
    $$B_n := A_n^{-1}M_{\Theta,n}A_{n-1} = \mathbb{I} + O(v^{-2})$$
    In consequence there exists limit $\alpha := (\prod_{m=n}^{\infty} B_m)\alpha_n$

$\Lambda > 0$: *the sketch of proof* (2)

- Convergence:
  - For large v ψ converges to certain function ψ^∞ – combination of $e_\pm$.
  - Due to form of B_n the corrections $\psi - \psi^\infty$ are normalizable.
  - Since limit itself is normalizable, all eigenfunctions are normalizable.
- Method works only for $\Lambda < \Lambda_c = 3\pi G/\beta$.
- For $\Lambda > \Lambda_c$ physical Hilbert space has at most 1 element.

Summary

- Evolution operator Θ is:
  - essentially self-adjoint for $\Lambda \leq 0$
  - not essentially self-adjoint for $\Lambda > 0$.
- For $\Lambda = 0$ the spectrum
  - $\mathrm{Sp}(\Theta) = [0, \infty)$ and
  - is absolutely continuous.
- For $\Lambda < 0$ (any k) or $\Lambda = 0, k = 1$, $k = 1$ spectrum is
  - discrete, and
  - for given $E > 0$ the number of eigenvalues s.t. $\lambda < E$ is bounded from above.
- For $\Lambda > 0$:
  - There is 4-dimensional family of self-adjoint extensions of Θ.
  - Spectrum of each extension is discrete.