LOOPS and FOAMS in 3 Dimensions

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References

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Many of us worked on closely related topics :

A. Ashtekar, W. Fairbairn, K. Krasnov, D. Oriti
J. Ryan, C. Rovelli, T. Thiemann, T. Tlas...

Motivations

Can SF and LQG reproduce known results of 3D quantum gravity ?

•Up to now, not Completely !

▷ We are going to recall what has been done in other approaches

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•Yes : coupling to matter fields ; geometry at Planck scale

▷ We are going to show the emergence of non-commutatity consequences on dynamics of fields

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Theory very interesting in itself : many aspects are not understood

▷ How to quantize the theory with non-degenerate metrics ?

▷ The Lorentzian sector is still poorly understood...

Overview

1. 3D gravity is a topological field theory

- *Quantum gravity and topological invariants*
- *Emergence of quantum groups*

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2. Spin Foams models vs. Loop Quantum Gravity

- *The Spin-Foam picture : the Ponzano-Regge model*
- *The Canonical picture*

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2. Spin Foams models vs. Loop Quantum Gravity

- *The Spin-Foam picture : the Ponzano-Regge model*
- *The Canonical picture*

3. Quantum 3D geometry: non-commutativity and fuzzy space

- *How quantum gravity makes the geometry non-commutative ?*
- *3D geometry is fuzzy at the Planck scale*
- *Example of dynamics in 3D quantum geometry*

3D gravity as a topological field theory

Quantum Gravity and topological invariants

•**Schematically, the problem of quantum gravity is to understand :**

$$\mathcal{Z}(M,\mathcal{O}) \;=\; \int \mathcal{D}g \; e^{iS[g]} \; \mathcal{O}[g]$$

▷ What is the right action $S[g]$ to consider ?
▷ How to compute explicitely the observables $\mathcal{O}$?
▷ How to give a meaning to the measure $\mathcal{D}g$?

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•By construction, the partition function is a topological invariant :

- ▷ It depends only on the topology of M and $\mathcal{O}$
- ▷ It depends on the boudary (∂M) data

3D gravity as a topological field theory

The revolution initiated by E. Witten (90)

•**Relation between topological invariants and QFT is clear in 3D :**

- ▷ Chern-Simons action with $G = SU(2)$

$$S_{CS}[A] = \frac{4\pi}{k}\int_M \langle A \wedge dA + \frac{1}{3} A \wedge A \wedge A \rangle$$

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$$\mathcal{Z}(S^3) = \sqrt{2}\frac{\sin(\frac{\pi}{k+2})}{\sqrt{k+2}}$$

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▷ Witten gives a sense to the measure $\mathcal{D}A$

$$\mathcal{Z}(S^3) = \sqrt{2}\frac{\sin(\frac{\pi}{k+2})}{\sqrt{k+2}}$$

▷ he regularizes the Wilson loops (framing) and computes physical quantities :

$$\frac{\mathcal{Z}(S^3, W^I_\gamma)}{\mathcal{Z}(S^3)} = \frac{\sin(\frac{d_I \pi}{k+2})}{\sin(\frac{\pi}{k+2})}$$

3D gravity as a topological field theory

The important role of quantum groups (Reshetikhin-Turaev 92)

•**The Quantum group** $U_q(su(2))$ **appears in the quantization :**

▷ The quantum parameter :

$$q \ = \ \exp(i2\pi/(k+2))$$

▷ Physical quantities are evaluation of spin-networks colored with quantum groups representations

▷ Physical Hilbert space is described in terms of spin-networks colored with $U_q(su(2))$ representations

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•**The need of a framing to define observables :**

▷ Observables are associated to ribbon graphes

3D gravity as a topological field theory

Topological invariants and 3D gravity (1) (Witten 89)

•**Pure gravity is a Chern-Simons theory :**

▷ In the first order formalism, gravity is a BF theory

$$S_P[e,\omega] = \int_M \langle e \wedge F(\omega) - \frac{\Lambda}{3} e \wedge e \wedge e \rangle$$

▷ The Chern-Simons connection A is related to (e, ω)

$$A = \omega J + f(\Lambda) e P$$

▷ In the Euclidean case, (J, P) are the generators of $\mathfrak{g}$:

$$isu(2) \;\; for \;\; \Lambda = 0, \quad so(3,1) \;\; for \;\; \Lambda < 0, \quad so(4) \;\; for \;\; \Lambda > 0$$

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- **Wilson loops are related to relativistic particles :**
  - ▷ Let R be a representation of $\mathfrak{g}$

$$S_R[x;A] = \int \langle C_R ; x^{-1} dx + x^{-1} A x \rangle$$

is the first order action for a free particle in the background A.

3D gravity as a topological field theory

Topological invariants and 3D gravity (2)

•**In the case of $ISU(2)$ for example**

- ▷ $C_R = mJ_0 + sP_0$ contains the mass and the spin
- ▷ $x = (u, q)$ where q is the position and $p_a = u_a^0$ is the momentum
- ▷ e.o.m : relativistic particle and $p^2 = m^2$
- ▷ In general (euclidean), e.o.m. are geodesic on H^3 or S^3

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- **The Wilson loop of A along γ is path int. of particle**

$$W_\gamma^R(A) = \int [\mathcal{D}x] \; e^{iS_R[x;A]}$$

  - ▷ This formula makes sense after reg. when R is finite dimensional

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•**The partition function of Chern-Simons theory**

$$\mathcal{Z}(M, C_R) = \int [\mathcal{D}A]\, W_\gamma^R(A)\, e^{iS_{CS}[A]}$$

is the path integral of 3D gravity coupled to particles

Summary and Questions

- **The work of Witten is Euclidean quantum gravity with** $\Lambda > 0$
- **Quantum groups are a result of the quantization**
- **Need of framing to define Wilson loops (they become ribbons)**

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- **The work of Witten is Euclidean quantum gravity with $\Lambda > 0$**
- **Quantum groups are a result of the quantization**
- **Need of framing to define Wilson loops (they become ribbons)**

- **Do SF models and LQG reproduce these results ?**
- **Can we do more ?**
- **Do that help to understand full theories ?**

Spin Foams models vs. Loop Quantum Gravity

Spin-Foam models (1) : Ponzano-Regge model (68)

•Computation of the path integral of pure gravity

- ▷ Euclidean signature with $\Lambda = 0$

$$\mathcal{Z}_{PR}(M) = \int [\mathcal{D}e][\mathcal{D}\omega] \exp(i \int_M e \wedge F(\omega)) = \int [\mathcal{D}\omega]\, \delta(F(\omega))$$

- ▷ Using a triangulation T of M to compute the path integral
- ▷ e leaves on edges of T and U on edges of T^* (dual complex)

$$\mathcal{Z}_{PR}(T) = \int \prod_{\ell \in T^*} dU_\ell \; \delta(\prod_{\ell \subset f \in T^*} U_\ell)$$

- ▷ Using Peter-Weyl and recoupling theory for $SU(2)$

$$\mathcal{Z}_{PR}(M) = \sum_{\{j_e\}} \prod_{e \in T} (2j_e + 1) \prod_{t \in T} (-)^{\sum_i j_i^t} \left\{ \begin{matrix} j_1^t & j_2^t & j_3^t \\ j_4^t & j_5^t & j_6^t \end{matrix} \right\}$$

- ▷ As expected, it is (almost) a topological invariant but formal
- ▷ It needs a gauge fixing

Spin Foams models vs. Loop Quantum Gravity

Spin-Foam models (2) : Inclusion of a cosmological constant (Turaev-Viro 92)

•Regularization by introducing a cosmological constant

▷ The Turaev-Viro model

$$\mathcal{Z}_{TV}(M) = K^{\#v} \sum_{\{j_e\}} \prod_{e\in T} [2j_e+1]_q \prod_{t\in T} (-)^{\sum_i j_i^t} \left\{ \begin{array}{ccc} j_1^t & j_2^t & j_3^t \\ j_4^t & j_5^t & j_6^t \end{array} \right\}_q$$

▷ Based on the quantum group $U_q(su(2))$ with $q = e^{2i\pi/(\sqrt{\Lambda}+2)}$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad K = -\frac{(q-q^{-1})^2}{2(\sqrt{\Lambda}+2)}$$

▷ It is supposed to come from the path integral of gravity

$$\mathcal{Z}_{TV}(M) \overset{?}{=} \int [\mathcal{D}e][\mathcal{D}\omega] \exp(i \int_M e \wedge F(\omega)) + \frac{\Lambda}{3} e \wedge e \wedge e)$$

Spin Foams models vs. Loop Quantum Gravity

Remarks concerning the TV Model

•Is the sum computable ?

▷ One can compute its value for S^3

▷ We choose a minimal triangulation of S^3 : one tetrahedron

$$\mathcal{Z}_{TV}(S^3) = K^4 \sum_{j_1 \cdots j_6} \prod_{i=1}^{6} [2j_i + 1]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_q^2 = \frac{2}{k+2} \sin^2\left(\frac{\pi}{k+2}\right)$$

▷ One can in principle compute topology changing amplitudes

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  - ▷ One can in principle compute topology changing amplitudes
- **The question of the classical limit**
  - ▷ $\ell_P \rightarrow 0$, $j \rightarrow \infty$ and $\ell_P j = \ell_j$

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_q \sim \cos(S_{EH}) + \cdots$$

  - ▷ Discrepencies between first order and second order gravity

Spin Foams models vs. Loop Quantum Gravity

Inclusion of particles (Freidel-Louapre-Livine and KN-Perez :04-...)

•**Introduction of particles in Euclidean case with** $\Lambda = 0$

- ▷ Particles are Wilson lines colored with (m, s) where $m \leq \pi/\ell_P$
- ▷ The mass introduce a curvature and the spin a torsion
- ▷ (m, s) are the representation of a quantum group $DSU(2)$
- ▷ For $s = 0$, the PR partition function is modified as follows :

$$\mathcal{Z}_{PR}(M, C_\gamma^m) = \sum_{\{j_e\}} \prod_{e \in T} (2j_e + 1) \prod_{\ell \in \gamma} \chi_{j_\ell}(m) \prod_{t \in T} (-)^{\sum_i j_i^t} \left\{ \begin{array}{ccc} j_1^t & j_2^t & j_3^t \\ j_4^t & j_5^t & j_6^t \end{array} \right\}$$

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- ▷ With gauge fixing (choice of graph), $\mathcal{Z}_{PR}$ is finite

•**Physical content of that formula**

- ▷ Expectation values of some observables
- ▷ With non trivial ∂M, $\mathcal{Z}_{PR}$ is the amplitude of topology changing or creation/annihilation of particles

Spin Foams models vs. Loop Quantum Gravity

Spin-Foam models (4) : Emergence of a non-commutativity

- **The calculation of the partition function with particles** :

$$\mathcal{Z}(M, C_\gamma^m) = \int [\mathcal{D}A][\mathcal{D}x] \exp(i(S_{CS}[A] + S_m[x; A]))$$

▷ We integrate x first : modified PR model

▷ What happens if we integrate A first ?

$$\mathcal{Z}(M, C_\gamma^m) = \int [\mathcal{D}x] \exp(iS_{eff}[x]) = \int [\mathcal{D}\varphi]\mathcal{O}[\varphi] \exp(iS_{eff}[\varphi])$$

▷ $S_{eff}[\phi]$ is a non-dynamical Group Field Theory

Spin Foams models vs. Loop Quantum Gravity

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- **Here the non-commutativity appears**

▷ One fixes the mass representations to one mass value m

▷ One adds a dynamics : ambiguities

but motivated by causality arguments (Oriti-Tlas)

Spin Foams models vs. Loop Quantum Gravity

Spin-Foam models (5) : Effective Field theory (Freidel-Livine and KN)

- **The effective field theory is a non-commutative QFT**

$$S_{eff}[\varphi] = \int d^3x \, (\partial_\mu \varphi \star \partial_\mu \varphi + \sin^2(m) \varphi \star \varphi)$$

▷ The field can be expanded in (grouplike) modes

$$\varphi(x) = \int dg \; \exp(iP(g) \cdot x) \, \varphi(x)$$

▷ The momenta are grouplike, defined with ambiguities
One choice being $P_a(g) = \mathrm{tr}(J_a g)$
▷ The space of momenta is curved ($SU(2)$) and compact
▷ As a result, the theory is free of UV divergences.

Spin Foams models vs. Loop Quantum Gravity

Spin-Foam models (6) : The non-commutativity

- **The $\star$ product is defined from the addition rule of momenta**
  - ▷ The addition rule of momenta is non-commutative

$$P(g_1) \oplus P(g_2) = P(g_1 g_2)$$

  - ▷ The non-commutative $\star$ product

$$\exp(iP(g_1)\cdot x) \star \exp(iP(g_2)\cdot x) = \exp(iP(g_1 g_2)\cdot x)$$

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- **Symmetry of the action**
  - ▷ Modified covariance $\delta_\xi S[\phi] \neq S[\delta_\xi \phi]$ for Leibnitz rule

$$\partial_a(\phi_1 \star \phi_2) = \partial_a \phi_1 \star \phi_2 + \phi_1 \star \partial_a \phi_2 - i\frac{\ell_P}{2}\epsilon_{ab}^{c}\partial_b\phi_1 \star \partial_c \phi_2 + \cdots$$

  - ▷ The symmetry group is $DSU(2)$

$$\delta_\xi S[\phi] = \delta_\xi \sum_n S^{(n)}[\phi \otimes \ldots \otimes \phi] = \sum_n S^{(n)}[\delta_\xi(\phi \otimes \ldots \otimes \phi)]$$

Spin Foams models vs. Loop Quantum Gravity

Loop Quantum gravity (1) : classical description (KN-Perez and KN)

The classical Hamiltonian theory is simple

▷ The Manifold is $M = \Sigma \times \mathbb{I}$

▷ The non-reduced phase space variables

$$\{e_a^i(x)\,,\,\omega_b^j(y)\} = \delta_{ab}\delta^{ij}\delta(x-y)$$

▷ The constraints of the theory

$$F(\omega) = \Lambda e \wedge e \;\; and \;\; T(e,\omega) = 0$$

They form a Lie algebra $\mathfrak{g}$

▷ The physical phase space is finite dimensional

▷ Non local dof, only topological dof

Spin Foams models vs. Loop Quantum Gravity

Loop Quantum gravity (2) : quantization à la Loop

- **The quantization follows the same lines as in 4D LQG**
  - ▷ Introduce the space of cylindrical functions
  - ▷ $\mathcal{H}_{kin}$ is described in terms of spin-networks
  - ▷ $\mathcal{H}_{phys}$ is defined by the projector P

$$\langle \phi, \psi \rangle_{phys} = \langle \phi, P\psi \rangle_{phys} = \mathcal{Z}_{PR}(M)$$

The link between SF and LQG is clear in 3D

Spin Foams models vs. Loop Quantum Gravity

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- **The physical Hilbert space and the Quantum Double**
  - ▷ $\mathcal{H}_{phys}$ is a representation space of $DSU(2)$
  - ▷ Parametrization of $\mathcal{H}_{phys}$ in terms of UIR of $DSU(2)$

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•**Possible Generalizations**

- ▷ Results Generalized when there are particles
- ▷ The case of cosmological constant is rather difficult

Spin Foams models vs. Loop Quantum Gravity

Loop Quantum gravity (3) : geometrical operators (Meusburger,KN)

- **Length operator at the kinematical level**
  - ▷ Defined by analogy with the area operator : smeared tetrad variable along a path
  - ▷ When the path crosses the edges of a spin-network : discrete contribution given by $\ell_P \sqrt{I(I+1)}$

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- **In the case of a torus for example**
  - ▷ Kinematical states on a minimal graph

$$\Phi_{IJK}(a,b) = \int dg \; \chi_I(ag) \, \chi_J(bg) \, \chi_K(g)$$

  - ▷ Length operator acts as derivative

$$D_a \Phi_{IJK}(a,b) = \partial_i(a) \partial^i(a) \Phi_{IJK}(a,b)$$

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- ▷ These states are no-longer eigenstates of length operator

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  - ▷ $DSU(2)$ is non-compact and representations are (m, s)
  - ▷ These states are no-longer eigenstates of length operator
- **In the case of the Torus**
  - ▷ Physical states are distributional functions

$$\chi_{m,n}(a,b) = \int dg\, \delta_a(gh(m)g^{-1})\, \delta_b(gh(n)g^{-1})$$

  - ▷ Orthonormal with the physical scalar product
  - ▷ Eigenstates of the "energy" operators $W(a)$ and $W(b)$
  - ▷ Eigenstates of D are obtained by FT

$$\tilde{\chi}_{I,J} = \int d\mu(m,n)\ \chi_I(h(m))\chi_J(h(n))\chi_{m,n}(a,b)$$

Spin Foams models vs. Loop Quantum Gravity

Loop Quantum gravity (3) : geometrical operators (Meusburger,KN)

- **Length operator at the physical level**
  - ▷ The physical states are constructed from UIR of $DSU(2)$
  - ▷ $DSU(2)$ is non-compact and representations are (m, s)
  - ▷ These states are no-longer eigenstates of length operator
- **In the case of the Torus**
  - ▷ Physical states are distributional functions

$$\chi_{m,n}(a,b) = \int dg \, \delta_a(gh(m)g^{-1}) \, \delta_b(gh(n)g^{-1})$$

  - ▷ Orthonormal with the physical scalar product
  - ▷ Eigenstates of the "energy" operators $W(a)$ and $W(b)$
  - ▷ Eigenstates of D are obtained by FT

$$\tilde{\chi}_{I,J} = \int d\mu(m,n) \, \chi_I(h(m))\chi_J(h(n))\chi_{m,n}(a,b)$$

Geometry remains discrete. But problem of interpretation ?

3D Quantum geometry : non-commutivity and fuzzy spaces

Deformation of the isometry group

•**The quantum double** $DSU(2)$

- The space of translations is curved due to gravity

$$DSU(2)\text{"} = \text{"}SU(2)\hat{\times}SU(2)^*$$

- It is a deformation of Euclidean group $ISU(2)$
- The Newton constant G is the deformation parameter

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  - ▷ It is the space of distribution on $SU(2)$ with the convolution

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  - ▷ The algebra of functions of momenta $\mathcal{A}$ is non-commutative
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- **The space is deformed**
  - ▷ Kind of Fourier transform : $\mathcal{A} \rightarrow \mathcal{F}(\mathbb{R}^{\ni})$

$$\Phi(x) = \int dg\, \phi(g)\, e^{iP(g)\cdot x}$$

  - ▷ Ambiguities in the definition of $P(g)$

3D Quantum geometry : non-commutivity and fuzzy spaces

Structure of the quantum geometry (Freidel-Majid and KN)

- **The space is fuzzy or discrete**
  - ▷ The Fourier transform of the space of momenta

$$\mathcal{A} \rightarrow \oplus_I \mathsf{Mat}_I(\mathbb{C})$$

  - ▷ The convolution product becomes the matrix product
  - ▷ Natural action of $DSU(2)$: rotations and translations

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  - ▷ Natural action of $DSU(2)$: rotations and translations
- **The geometry of the space**
  - ▷ Coordinates are non-commutative

$$[x_a, x_b] = \epsilon_{ab}^{c} x_c \;\; \to \;\; x_a = \ell_P J_a$$

  - ▷ Delta distribution is spread :

no localisation of space points

3D Quantum geometry : non-commutivity and fuzzy spaces

Consequences on dynamics

•**Effects on Quantum Field Theories**

- ▷ Resolution of UV divergences : the momenta are bounded
- ▷ Modifications of pairs creation rate

3D Quantum geometry : non-commutivity and fuzzy spaces

Consequences on dynamics

- **Effects on Quantum Field Theories**
  - ▷ Resolution of UV divergences : the momenta are bounded
  - ▷ Modifications of pairs creation rate
- **The dynamics becomes fundamentaly discrete**
  - ▷ The Field is described as a family of matrices
  - ▷ Derivatives are replaced by finite diff. operators

$$(\partial_a M)^j_{st} = \frac{2i}{\ell_P d_j} D^{1/2}_{pq}(J_a) \quad \Big(\sqrt{(j+1+2qs)(j+1+2tp)}\; M^{j+1/2}_{q+s\, p+t} + (-1)^{q-p}\sqrt{(j-2qs)(j-2pt)}\; M^{j-1/2}_{q+s\, p+t}\Big)$$

  - ▷ Prediction of quantum gravity effects

3D Quantum geometry : non-commutivity and fuzzy spaces

Example : dynamics of a particle (KN-08)

- The momentum representation : the field is $\phi(\theta)$
- The matrix representation :

$$\phi_a = \frac{1}{2\pi} \int d\theta e^{ia\theta} \phi(\theta)$$

- The continuous representation

$$\phi(t) = \sum_a J_a(t/\ell_P)\phi_a$$

- The continuous representation depends on all the microscopic time a

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- ▷ The continuous representation depends on all the microscopic time *a*
- ▷ When the dynamics is linear : $\phi(t)$ coincides with $\phi_{cl}(t)$
- ▷ When the dynamics is non-linear : deviation from the classical theory !

Discussion

•We can go very far in 3D Quantum gravity

▷ Precise description of $\mathcal{H}_{phys}$

▷ Link between SF anf LQG is very clear

▷ Precise description of the quantum geometry

▷ Background independent formulation of dynamics

Discussion

•We can go very far in 3D Quantum gravity

▷ Precise description of $\mathcal{H}_{phys}$

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•Many interesting questions

▷ The question of the cosmological constant

▷ Study the Lorentzian sector

▷ Precise construction of coherent states within SF or LQG

▷ LQG description of Black holes

▷ Recover 3D quantum gravity from 4D LQG...