Doubly Special Relativity from Quantum Gravity

Jurek Kowalski-Glikman

Zakopane QG Workshop, 2008

The papers

- Laurent & Artem "Quantum gravity in terms of topological observables," arXiv:hep-th/0501191
- Laurent, Artem & JKG "Particles as Wilson lines of gravitational field," Phys. Rev. D **74**, 084002 (2006) (arXiv:gr-qc/0607014)
- Artem & JKG "Can we see gravitational collapse in (quantum) gravity perturbation theory?," arXiv:gr-qc/0612093 and to appear
- With thanks to Florian Girelli, Catherine Meusburger, and Bernd Schroers

Outline

Outline

1. What is DSR (briefly)?
2. Gravity as a constrained BF theory
3. Perturbation theory
4. Chern-Simons and DSR

Outline

1. What is DSR (briefly)?
2. Gravity as a constrained BF theory
3. Perturbation theory
4. Chern-Simons and DSR

Outline

What is DSR?

- The DSR problem: Are there any Planck scale corrections to flat space particles kinematics?
- If there are some, they must be characterized by presence of the Planck scale in description of kinematics (e.g. modified conservation laws, deformed dispersion relations, deformed transformations between inertial observers, etc.)
- If DSR is real, it is expected to lead to rich phenomenology.

What is DSR?

- The DSR problem: Are there any Planck scale corrections to flat space particles kinematics?
- If there are some, they must be characterized by presence of the Planck scale in description of kinematics (e.g. modified conservation laws, deformed dispersion relations, deformed transformations between inertial observers, etc.)
- If DSR is real, it is expected to lead to rich phenomenology.

What is DSR?

- The DSR problem: Are there any Planck scale corrections to flat space particles kinematics?
- If there are some, they must be characterized by presence of the Planck scale in description of kinematics (e.g. modified conservation laws, deformed dispersion relations, deformed transformations between inertial observers, etc.)
- If DSR is real, it is expected to lead to rich phenomenology.

What is DSR?

- In DSR one assumes that (contrary to Lorentz symmetry breaking schemes), in spite of the presence of the additional scale, relativity principle still holds.
- It follows that relativistic symmetries must be deformed, so as to make the new scale observer-independent. One way of achieving this is to assume that relativistic symmetries are described by quantum groups, with dimensionful deformation parameter, κ, of order of Planck mass.
- But then, of course, DSR cannot be fundamental – it must emerge from some more fundamental theory of quantum space, time, and particles (fields) in an appropriate limit.

What is DSR?

- In DSR one assumes that (contrary to Lorentz symmetry breaking schemes), in spite of the presence of the additional scale, relativity principle still holds.
- It follows that relativistic symmetries must be deformed, so as to make the new scale observer-independent. One way of achieving this is to assume that relativistic symmetries are described by quantum groups, with dimensionful deformation parameter, κ, of order of Planck mass.
- But then, of course, DSR cannot be fundamental – it must emerge from some more fundamental theory of quantum space, time, and particles (fields) in an appropriate limit.

What is DSR?

- In DSR one assumes that (contrary to Lorentz symmetry breaking schemes), in spite of the presence of the additional scale, relativity principle still holds.
- It follows that relativistic symmetries must be deformed, so as to make the new scale observer-independent. One way of achieving this is to assume that relativistic symmetries are described by quantum groups, with dimensionful deformation parameter, κ, of order of Planck mass.
- But then, of course, DSR cannot be fundamental – it must emerge from some more fundamental theory of quantum space, time, and particles (fields) in an appropriate limit.

DSR and Quantum Gravity

- The name of this fundamental theory is known for decades – it is Quantum Gravity.
- We should therefore quantize gravity coupled to particles, take the limit, in which all local degrees of freedom of gravity are switched off, and see what we get as a result.
- Then either some trace of gravity remains (so that the scale of quantum gravity is still present) – in which case we get DSR – or it does not – and we end up with standard SR.

DSR and Quantum Gravity

- The name of this fundamental theory is known for decades – it is Quantum Gravity.
- We should therefore quantize gravity coupled to particles, take the limit, in which all local degrees of freedom of gravity are switched off, and see what we get as a result.
- Then either some trace of gravity remains (so that the scale of quantum gravity is still present) – in which case we get DSR – or it does not – and we end up with standard SR.

DSR and Quantum Gravity

- The name of this fundamental theory is known for decades – it is Quantum Gravity.
- We should therefore quantize gravity coupled to particles, take the limit, in which all local degrees of freedom of gravity are switched off, and see what we get as a result.
- Then either some trace of gravity remains (so that the scale of quantum gravity is still present) – in which case we get DSR – or it does not – and we end up with standard SR.

The DSR project

- Step 1.

Find a formulation of gravity in which switching off local gravitational degrees of freedom makes sense.

- ✓

Mac Dowell & Mansouri formulation ("gravity as a constrained BF theory"). Gravity is formulated as a $SO(4,1)$ BF theory with constraint breaking the symmetry down to $SO(3,1)$. In the limit in which the constraint goes away, we have to do with topological theory, with no local degrees of freedom of gravity.

Smolin, Freidel, Starodubtsev

The DSR project

- Step 1.

Find a formulation of gravity in which switching off local gravitational degrees of freedom makes sense.

- ✓

Mac Dowell & Mansouri formulation (“gravity as a constrained BF theory”). Gravity is formulated as a $SO(4,1)$ BF theory with constraint breaking the symmetry down to $SO(3,1)$. In the limit in which the constraint goes away, we have to do with topological theory, with no local degrees of freedom of gravity.

Smolin, Freidel, Starodubtsev

The DSR project

- Step 2. ✓

Particle coupling – done for any mass and spin. Particles are Wilson lines of gravitational field.

Freidel, Starodubtsev, JKG

The DSR project

- Step 3.

Quantum perturbation theory. The gauge breaking term comes with coupling constant α, which is very small. One can formulate the quantum (path integral) perturbation theory in α. In particular the zeroth order of this perturbation theory will lead to (quantum theory) of particles on the background of quantum spacetime with no local gravitational degrees of freedom.

Freidel, Starodubtsev, JKG

The DSR project

- Step 4.

Quantum gravitational holography. In this limit the theory reduces to Chern-Simons $SO(4,1)$ theory on the boundary of 4d spacetime manifold. The particle(s) worldline(s) is forced to live on the boundary as well.

Starodubtsev, JKG

The DSR project

- Step 5.

Chern-Simons theory with particle(s) is equivalent to effective theory of deformed particle(s) kinematics. This deformation depends of two scales: the ultraviolet Planck scale κ and the infrared cosmological constant scale Λ. In the limit $\Lambda \to 0$ we end up with deformed particle(s) kinematics with just single scale κ. This is nothing but DSR!

Starodubtsev, JKG

3d and 4d.

- Gravity in 3d is described by topological field theory (and thus no local degrees of freedom of gravitational field.)

$$S = \int e \wedge F(\omega), \quad \text{or} \quad S = \int A \wedge dA + A^3$$

- If we couple gravity to particles/fields we get an effective DSR theory
- Can something similar be done in 4d?
- Yes!

3d and 4d.

- Gravity in 3d is described by topological field theory (and thus no local degrees of freedom of gravitational field.)

$$S = \int e \wedge F(\omega), \quad \text{or} \quad S = \int A \wedge dA + A^3$$

- If we couple gravity to particles/fields we get an effective DSR theory
- Can something similar be done in 4d?
- Yes!

3d and 4d.

- Gravity in 3d is described by topological field theory (and thus no local degrees of freedom of gravitational field.)

$$S = \int e \wedge F(\omega), \quad \text{or} \quad S = \int A \wedge dA + A^3$$

- If we couple gravity to particles/fields we get an effective DSR theory
- Can something similar be done in 4d?
- Yes!

Gravity as a constrained BF theory.

- In 4d gravity is certainly not topological! But it can be described as "TFT" + "small perturbation"
- Take BF theory of SO(4, 1) (de Sitter) group

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ}$$

- Add gauge breaking term, which breaks the symmetry down to SO(3, 1)

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- This theory is equivalent to GR! (up to topological terms)

Gravity as a constrained BF theory.

- In 4d gravity is certainly not topological! But it can be described as "TFT" + "small perturbation"
- Take BF theory of SO(4, 1) (de Sitter) group

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ}$$

- Add gauge breaking term, which breaks the symmetry down to SO(3, 1)

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- This theory is equivalent to GR! (up to topological terms)

Gravity as a constrained BF theory.

- In 4d gravity is certainly not topological! But it can be described as "TFT" + "small perturbation"
- Take BF theory of SO(4, 1) (de Sitter) group

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ}$$

- Add gauge breaking term, which breaks the symmetry down to SO(3, 1)

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- This theory is equivalent to GR! (up to topological terms)

Gravity as a constrained BF theory.

- In 4d gravity is certainly not topological! But it can be described as "TFT" + "small perturbation"
- Take BF theory of SO(4, 1) (de Sitter) group

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ}$$

- Add gauge breaking term, which breaks the symmetry down to SO(3, 1)

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- This theory is equivalent to GR! (up to topological terms)

Equivalence with GR

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- We decompose the SO(4, 1) connection

$$A_\mu{}^{a4} = \frac{1}{2l} e_\mu{}^a, \quad A_\mu{}^{ab} = \omega_\mu{}^{ab}$$

- take appropriate values of the parameters

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G\Lambda}{3} \frac{1}{(1+\gamma^2)}, \quad \beta = \frac{G\Lambda}{3} \frac{\gamma}{(1+\gamma^2)}$$

- and solve for B

Equivalence with GR

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- We decompose the SO(4, 1) connection

$$A_\mu{}^{a4} = \frac{1}{2l} e_\mu{}^a, \quad A_\mu{}^{ab} = \omega_\mu{}^{ab}$$

- take appropriate values of the parameters

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G\Lambda}{3} \frac{1}{(1+\gamma^2)}, \quad \beta = \frac{G\Lambda}{3} \frac{\gamma}{(1+\gamma^2)}$$

- and solve for B

Equivalence with GR

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- We decompose the SO(4, 1) connection

$$A_\mu{}^{a4} = \frac{1}{2l} e_\mu{}^a, \quad A_\mu{}^{ab} = \omega_\mu{}^{ab}$$

- take appropriate values of the parameters

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G\Lambda}{3} \frac{1}{(1+\gamma^2)}, \quad \beta = \frac{G\Lambda}{3} \frac{\gamma}{(1+\gamma^2)}$$

- and solve for B

Equivalence with GR

- As a result we get

$$S = \widetilde{S}_P - \frac{\alpha}{4(\alpha^2+\beta^2)} \int R^{ij}(\omega) \wedge R^{kl}(\omega)\epsilon_{ijkl}$$
$$+ \frac{\beta}{2(\alpha^2+\beta^2)} \int R^{ij}(\omega) \wedge R_{ij}(\omega) + \frac{1}{\beta}\int C$$

- The last three terms are the integrals of Euler, Pontryagin, and Nieh-Yan classes.

Equivalence with GR

- As a result we get

$$S = \widetilde{S}_P - \frac{\alpha}{4(\alpha^2+\beta^2)}\int R^{ij}(\omega)\wedge R^{kl}(\omega)\epsilon_{ijkl} + \frac{\beta}{2(\alpha^2+\beta^2)}\int R^{ij}(\omega)\wedge R_{ij}(\omega) + \frac{1}{\beta}\int C$$

- The last three terms are the integrals of Euler, Pontryagin, and Nieh-Yan classes.

Equivalence with GR

- The first term is the action of GR with cosmological constant and Immirzi parameter γ.

$$\widetilde{S}_P = \frac{1}{2G}\int R^{ij}(\omega)\wedge e^k \wedge e^l \epsilon_{ijkl}$$

$$-\frac{\Lambda}{12G}\int e^i \wedge e^j \wedge e^k \wedge e^l \epsilon_{ijkl} + \frac{1}{G\gamma}\int R^{ij}(\omega)\wedge e_i \wedge e_j$$

- Even if the term proportional to γ^{-1} is not topological (its variation is non zero), it doesn't affect the classical equation of motion when $\gamma^2 \neq 1$ unless the theory is coupled to fermions

Where could we go from here?

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- The α parameter is small and all terms in the action are manifestly diff-invariant. Thus we can try to do perturbations in α, both classical and quantum. In particular quantum perturbation theory will be manifestly diff-invariant.
- In $\alpha \to 0$ limit, the theory becomes topological. The hope is that in this limit (when local degrees of freedom of gravitational field are switched off), after coupling to particles/fields we recover DSR, as in 3d.

Where could we go from here?

$$S = \int B_{IJ} \wedge F^{IJ}(A) - \frac{\beta}{2} \int B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \int \epsilon^{IJKL4} B_{IJ} \wedge B_{KL}$$

- The α parameter is small and all terms in the action are manifestly diff-invariant. Thus we can try to do perturbations in α, both classical and quantum. In particular quantum perturbation theory will be manifestly diff-invariant.
- In $\alpha \to 0$ limit, the theory becomes topological. The hope is that in this limit (when local degrees of freedom of gravitational field are switched off), after coupling to particles/fields we recover DSR, as in 3d.

Particles as Wilson lines

- Matter can arise as breaking the gauge symmetry in a localized way. The gauge degrees of freedom are promoted to dynamical degree of freedom and reproduce the dynamics of a relativistic particle coupled to gravity. This realizes explicitly in 4d the idea that matter (relativistic particles) can arise as a charged (under $SO(4,1)$) topological gravitational defect.
- Equivalently, one observes that the only natural way to couple a gauge field to localized excitation is by insertions of Wilson lines. Remarkably, the dynamics of these Wilson lines is the one of relativistic particles.

Particles as Wilson lines

- Matter can arise as breaking the gauge symmetry in a localized way. The gauge degrees of freedom are promoted to dynamical degree of freedom and reproduce the dynamics of a relativistic particle coupled to gravity. This realizes explicitly in 4d the idea that matter (relativistic particles) can arise as a charged (under $SO(4,1)$) topological gravitational defect.
- Equivalently, one observes that the only natural way to couple a gauge field to localized excitation is by insertions of Wilson lines. Remarkably, the dynamics of these Wilson lines is the one of relativistic particles.

Particle in gravitational field

- The simplest possible localized gauge breaking coupling to the gravitational field A is obtained by choosing a worldline *P* and a fixed element *K* of the so(4, 1) Lie algebra

$$S_P(A) = -\int d\tau \, \mathrm{Tr}\, \left(K\, A_\tau(\tau)\right)$$

where τ parameterizes the world line $z^\mu(\tau)$ and $A_\tau(\tau) \equiv A_\mu(z(\tau))\, \dot{z}^\mu$.

- This action breaks both gauge invariance and diffeomorphism symmetry along the worldline.

Particle in gravitational field

- The simplest possible localized gauge breaking coupling to the gravitational field A is obtained by choosing a worldline *P* and a fixed element *K* of the so(4, 1) Lie algebra

$$S_P(A) = -\int d\tau \, \mathrm{Tr}\, \left(K A_\tau(\tau)\right)$$

where τ parameterizes the world line $z^\mu(\tau)$ and $A_\tau(\tau) \equiv A_\mu(z(\tau))\, \dot{z}^\mu$.

- This action breaks both gauge invariance and diffeomorphism symmetry along the worldline.

The particle action

- de Sitter group acts by conjugation on its algebra; the orbits are labeled by two numbers (m, s): the mass and spin of the particle. For each orbit we choose a fixed representative element of so(4, 1)

$$\mathsf{K} \equiv m l T^{04}/2 + s T^{23}/2$$

- Lagrangian of a particle in gravitational field is given by an embedding of its worldline $z(\tau)$ and a function $\mathsf{h}(\tau)$ in the Lorentz subgroup, representing Lorentz transformation from the rest frame, in which the Poincaré charges of the particle are described by K to an actual frame, in which the particle has momentum p_a and spin s_{ab}

The particle action

- de Sitter group acts by conjugation on its algebra; the orbits are labeled by two numbers (m, s): the mass and spin of the particle. For each orbit we choose a fixed representative element of so(4, 1)

$$\mathsf{K} \equiv m l T^{04}/2 + s T^{23}/2$$

- Lagrangian of a particle in gravitational field is given by an embedding of its worldline $z(\tau)$ and a function $\mathsf{h}(\tau)$ in the Lorentz subgroup, representing Lorentz transformation from the rest frame, in which the Poincaré charges of the particle are described by K to an actual frame, in which the particle has momentum p_a and spin s_{ab}

The particle action

- In terms of $A^h = h^{-1}Ah + h^{-1}dh$, lagrangian takes the simple form

$$L(z, h; A) = -\mathrm{Tr}\left(KA^h_\tau(\tau)\right) \qquad S = \int d\tau\, L(z, h; A)$$

Results

- Equations of motion of particle(s) in gravitational field reproduce the spin precession equation

$$D_\tau J^{ab} = \nabla_\tau s^{ab} + e_\tau{}^a p^b - e_\tau{}^b p^a = 0$$

$$\nabla_\tau \equiv \frac{d}{d\tau} + [\omega_\tau, \cdot]$$

- and Mathisson–Papapetrou equation (in the presence of torsion.)

$$(\nabla_\tau p_a) e_\mu{}^a = \frac{1}{2} s_{ab}\, R_{\mu\nu}{}^{ab}\, \dot{z}^\nu + p_a T_{\mu\nu}{}^a\, \dot{z}^\nu$$

Short summary

- For $\alpha \neq 0$ the constrained BF theory is dynamical gravity. In the $\alpha \to 0$ limit we get topological theory with no local degrees of freedom (and thus no gravitational waves, no static potential, etc.)
- The parameter β is related to Immirzi parameter; $\beta = 0$ corresponds to $\gamma = 0$.
- We know how to couple gravity in this formulation to point particles.

Short summary

- For $\alpha \neq 0$ the constrained BF theory is dynamical gravity. In the $\alpha \to 0$ limit we get topological theory with no local degrees of freedom (and thus no gravitational waves, no static potential, etc.)
- The parameter β is related to Immirzi parameter; $\beta = 0$ corresponds to $\gamma = 0$.
- We know how to couple gravity in this formulation to point particles.

Short summary

- For $\alpha \neq 0$ the constrained BF theory is dynamical gravity. In the $\alpha \to 0$ limit we get topological theory with no local degrees of freedom (and thus no gravitational waves, no static potential, etc.)
- The parameter β is related to Immirzi parameter; $\beta = 0$ corresponds to $\gamma = 0$.
- We know how to couple gravity in this formulation to point particles.

Perturbation theory: the setting

- The field equations for gravity coupled to point particle at rest at the origin

$$F^{IJ} = \alpha\, \epsilon^{IJKLM}\, B_{JK}\, v_M + \beta B^{IJ}$$
$$\mathcal{D}_A\, B^{IJ} = J^{IJ}\, \delta_P, \quad \delta_P = \delta^3(x)\varepsilon$$

  where J belongs to the conjugacy class of $K = m\ell T^{04} + sT^{23}$.

- Naively one would think that to get zero order approximation on should just set $\alpha = 0$. But this is not the case. There are also integrability conditions of higher order equations. They lead to full Einstein equations already at zeroth order.

Perturbation theory: the setting

- The field equations for gravity coupled to point particle at rest at the origin

$$F^{IJ} = \alpha\, \epsilon^{IJKLM}\, B_{JK}\, v_M + \beta B^{IJ}$$

$$\mathcal{D}_A\, B^{IJ} = J^{IJ}\, \delta_P, \quad \delta_P = \delta^3(x)\varepsilon$$

where J belongs to the conjugacy class of $K = m\ell T^{04} + sT^{23}$.

- Naively one would think that to get zero order approximation on should just set $\alpha = 0$. But this is not the case. There are also integrability conditions of higher order equations. They lead to full Einstein equations already at zeroth order.

However ...

- There is still something very interesting that can be said. Consider the field equations

$$F^{IJ}(A^{(0)}) = \beta B^{(0)\,IJ}, \quad \mathcal{D}_{A^{(0)}} B^{(0)\,IJ} = J^{IJ}$$

- As a result of Bianchi identity, these equations cannot be solved for non-trivial source if the connection is nonsingular. They can be solved however assuming that connection contains the part proportional to Dirac monopole connection

$$A = g^{-1}dg + \beta g^{-1}(A_D + \Phi)g, \quad A_D \sim a_D \equiv (1 - \cos\theta)d\phi$$

($g \in SO(4,1)$, Φ is a $so(4,1)$-valued 1-form.)

However ...

- There is still something very interesting that can be said. Consider the field equations

$$F^{IJ}(\mathrm{A}^{(0)}) = \beta B^{(0)\,IJ}, \quad \mathcal{D}_{\mathrm{A}^{(0)}}\, B^{(0)\,IJ} = J^{IJ}$$

- As a result of Bianchi identity, these equations cannot be solved for non-trivial source if the connection is nonsingular. They can be solved however assuming that connection contains the part proportional to Dirac monopole connection

$$A = \mathrm{g}^{-1}d\mathrm{g} + \beta \mathrm{g}^{-1}(A_D + \Phi)\mathrm{g}, \quad A_D \sim a_D \equiv (1 - \cos\theta)d\phi$$

($g \in SO(4,1)$, Φ is a $so(4,1)$-valued 1-form.)

The solution

- Now we must find a solution of Einstein equations with such connection. Such solution exists and is well known: it is de Sitter – Taub – NUT solution

$$ds^2 = -F(r)\,(dt + 2N(1-\cos\theta)d\phi)^2 + F(r)^{-1}\,dr^2$$
$$+\left(r^2+N^2\right)(d\theta^2 + \sin^2\theta\,d\phi^2)$$
$$F(r) = 1 - \frac{r^2}{l^2} - \frac{2N^2}{r^2+N^2}$$

- This is a solution describing static point mass (the Taub–NUT charge N is proportional to the mass) in the leading order of perturbation theory
- This solution has string-like singularity.

The solution

- Now we must find a solution of Einstein equations with such connection. Such solution exists and is well known: it is de Sitter – Taub – NUT solution

$$ds^2 = -F(r)\,(dt + 2N(1-\cos\theta)d\phi)^2 + F(r)^{-1}\,dr^2$$
$$+\left(r^2 + N^2\right)(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$F(r) = 1 - \frac{r^2}{l^2} - \frac{2N^2}{r^2 + N^2}$$

- This is a solution describing static point mass (the Taub–NUT charge N is proportional to the mass) in the leading order of perturbation theory
- This solution has string-like singularity.

The solution

- Now we must find a solution of Einstein equations with such connection. Such solution exists and is well known: it is de Sitter – Taub – NUT solution

$$ds^2 = -F(r)\,(dt + 2N(1-\cos\theta)d\phi)^2 + F(r)^{-1}\,dr^2$$
$$+\left(r^2 + N^2\right)(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$F(r) = 1 - \frac{r^2}{l^2} - \frac{2N^2}{r^2 + N^2}$$

- This is a solution describing static point mass (the Taub–NUT charge N is proportional to the mass) in the leading order of perturbation theory
- This solution has string-like singularity.

Alternatively

- One can couple gravity to both particles and strings (since the B field is a two-form it naturally couples to one dimensional objects.)
- In the case of a semi-infinite string and a particle, one can show that as a result of consistency of field equations the particle and the string endpoint merge into one (the same charge, the same position.)

Alternatively

- One can couple gravity to both particles and strings (since the B field is a two-form it naturally couples to one dimensional objects.)
- In the case of a semi-infinite string and a particle, one can show that as a result of consistency of field equations the particle and the string endpoint merge into one (the same charge, the same position.)

Quantum perturbation theory

- When we are interested in scattering of a finite number of particles in a finite order of perturbation theory we have to calculate Feynman diagrams, which are finite dimensional integrals instead of infinite dimensional path integrals.
- Thus quantum theory allows us to select a set of questions which could be answered by taking into account only finite number of modes of quantum field.
- This option is not available in classical perturbation theory where we have to solve infinite-dimensional equations in every order. This is a big advantage of quantum perturbation theory over the classical one.

Quantum perturbation theory

- When we are interested in scattering of a finite number of particles in a finite order of perturbation theory we have to calculate Feynman diagrams, which are finite dimensional integrals instead of infinite dimensional path integrals.
- Thus quantum theory allows us to select a set of questions which could be answered by taking into account only finite number of modes of quantum field.
- This option is not available in classical perturbation theory where we have to solve infinite-dimensional equations in every order. This is a big advantage of quantum perturbation theory over the classical one.

Quantum perturbation theory

- When we are interested in scattering of a finite number of particles in a finite order of perturbation theory we have to calculate Feynman diagrams, which are finite dimensional integrals instead of infinite dimensional path integrals.
- Thus quantum theory allows us to select a set of questions which could be answered by taking into account only finite number of modes of quantum field.
- This option is not available in classical perturbation theory where we have to solve infinite-dimensional equations in every order. This is a big advantage of quantum perturbation theory over the classical one.

The perturbative expansion

- The perturbative expression for the partition function coupled to arbitrary finite number of particles after integrating out B-field looks like

$$Z(\{x_{p_i}\},\{x_{p_f}\}) = \int \mathcal{D}A \sum_n \frac{(i\alpha)^n}{\beta^2 n!}\Big(\int \epsilon^{ijkl} B_{ij}(x)\wedge B_{kl}(x)\Big)^n$$

$$\times \exp\left[i\int_M L_{BF} + \sum S_p(x_{p_i}, x_{p_f})\right]$$

- Here S_p are actions for particles, and x_{p_i} and x_{p_f} are initial and final points of p-th particle.

The perturbative expansion

- The perturbative expression for the partition function coupled to arbitrary finite number of particles after integrating out B-field looks like

$$Z(\{x_{p_i}\},\{x_{p_f}\}) = \int \mathcal{D}A \sum_n \frac{(i\alpha)^n}{\beta^2 n!}\left(\int \epsilon^{ijkl} B_{ij}(x)\wedge B_{kl}(x)\right)^n$$

$$\times \exp\left[i\int_M L_{BF} + \sum S_p(x_{p_i}, x_{p_f})\right]$$

- Here S_p are actions for particles, and x_{p_i} and x_{p_f} are initial and final points of p-th particle.

The zeroth order of perturbative expansion

- In the 0th order of perturbative expansion of QG local degrees of freedom of gravity are not present. There are just particles on topological background. This is the DSR limit of quantum gravity.
- The zeroth order partition function has the form

$$Z(\{x_{p_i}\},\{x_{p_f}\}) = \int \mathcal{D}A \exp\left[i\int_M L\right]$$

- where

$$L = \frac{1}{\beta}F^{AB}\wedge F_{AB} + K_p^{AB}\, A^h_{0\,AB}\,\delta^3(x-x_p), \qquad K_p = \mu_p T^{04}$$

The zeroth order of perturbative expansion

- In the 0th order of perturbative expansion of QG local degrees of freedom of gravity are not present. There are just particles on topological background. This is the DSR limit of quantum gravity.
- The zeroth order partition function has the form

$$Z(\{x_{p_i}\},\{x_{p_f}\}) = \int \mathcal{D}A \exp\left[i\int_M L\right]$$

- where

$$L = \frac{1}{\beta} F^{AB} \wedge F_{AB} + K_p^{AB} A^h_{0\,AB}\, \delta^3(x - x_p), \quad K_p = \mu_p T^{04}$$

Holography

- The integral of the first term in the action reduces to Chern-Simons action on the boundary of the original manifold $\mathcal{M}$

$$\frac{1}{\beta}\int_{\mathcal{M}} F^{AB}\wedge F_{AB} = \frac{1}{\beta}\int_{\partial\mathcal{M}} Y_{CS}(A)$$

- The integral of the second term has the form

$$\int_{\mathcal{M}} Tr(A^h K)\,\delta^3(x - x_p)$$

- and will contribute to path integral only if the particle worldline belongs to the boundary. It can be shown that particle is indeed forced to live on the boundary.

Holography

- The integral of the first term in the action reduces to Chern-Simons action on the boundary of the original manifold $\mathcal{M}$

$$\frac{1}{\beta}\int_{\mathcal{M}} F^{AB}\wedge F_{AB} = \frac{1}{\beta}\int_{\partial\mathcal{M}} Y_{CS}(A)$$

- The integral of the second term has the form

$$\int_{\mathcal{M}} Tr(A^h K)\,\delta^3(x-x_p)$$

- and will contribute to path integral only if the particle worldline belongs to the boundary. It can be shown that particle is indeed forced to live on the boundary.

Holography

- Thus the zeroth order of perturbation theory is described by quantum Chern-Simons theory coupled to particle(s) on the S^2_n boundary of the original manifold $\mathcal{M}$, given by the action

$$S = \int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi} \langle \partial_0 A_S \wedge A_S \rangle - \int_{\mathbb{R}} dx^0 \sum_{p=1}^{n} \left\langle D_p \, , \, h_p^{-1} \partial_0 h_p \right\rangle$$

$$\int_{\mathbb{R}} dx^0 \int_{S^2_n} \left\langle A_0 \, , \, \frac{k}{2\pi} F_S - \sum_{p=1}^{n} T_p \delta^{(2)}(x - x_{(p)}) dx^1 \wedge dx^2 \right\rangle$$

- Strong holography: perhaps this holds in all orders of perturbation theory.

Holography

- Thus the zeroth order of perturbation theory is described by quantum Chern-Simons theory coupled to particle(s) on the S^2_n boundary of the original manifold $\mathcal{M}$, given by the action

$$S = \int_{\mathbb{R}} dx^0 \int_{S^2_n} \frac{k}{4\pi} \langle \partial_0 A_S \wedge A_S \rangle - \int_{\mathbb{R}} dx^0 \sum_{p=1}^{n} \left\langle D_p \, , \, h_p^{-1} \partial_0 h_p \right\rangle$$

$$\int_{\mathbb{R}} dx^0 \int_{S^2_n} \left\langle A_0 \, , \, \frac{k}{2\pi} F_S - \sum_{p=1}^{n} T_p \delta^{(2)}(x - x_{(p)}) dx^1 \wedge dx^2 \right\rangle$$

- Strong holography: perhaps this holds in all orders of perturbation theory.

CS theory with particles

- Away from the particle(s) worldline(s) CS connection is pure gauge. All the information about particles is encoded in holonomies.
- In other words, gauge degrees of freedom become dynamical at the location(s) of the particle(s). They effectively deform particle(s) kinematics.

CS theory with particles

- Away from the particle(s) worldline(s) CS connection is pure gauge. All the information about particles is encoded in holonomies.
- In other words, gauge degrees of freedom become dynamical at the location(s) of the particle(s). They effectively deform particle(s) kinematics.

Alekseev/Malkin & Fock/Rosly

- The effective theory of particle(s) is described by powerful theorems of Alekseev & Malkin (symplectic structure) and Fock & Rosly (Poisson bivector).
- This structures are fully encoded in gauge group structure.

Alekseev/Malkin & Fock/Rosly

- The effective theory of particle(s) is described by powerful theorems of Alekseev & Malkin (symplectic structure) and Fock & Rosly (Poisson bivector).
- This structures are fully encoded in gauge group structure.

Fock & Rosly bivector

- The Fock & Rosly bivector makes it possible to calculate the Poisson brackets on the phase space of the particles.

$$B_{FR} = \frac{1}{2}\, r^{ab} \left(X^R_a \wedge X^R_b + X^L_a \wedge X^L_b \right) + r^{ab}\, X^R_a \wedge X^L_b$$

- Here r^{ab} is the classical r matrix of the gauge group, whose symmetric part is equal to the gauge group Casimir, used to define the product $< \star, \star >$ in the CS action; $X^{R/L}_a$ are right/left invariant vector fields on the gauge group.

Fock & Rosly bivector

- The Fock & Rosly bivector makes it possible to calculate the Poisson brackets on the phase space of the particles.

$$B_{FR} = \frac{1}{2}\, r^{ab} \left(X_a^R \wedge X_b^R + X_a^L \wedge X_b^L \right) + r^{ab}\, X_a^R \wedge X_b^L$$

- Here r^{ab} is the classical r matrix of the gauge group, whose symmetric part is equal to the gauge group Casimir, used to define the product $< \star, \star >$ in the CS action; $X_a^{R/L}$ are right/left invariant vector fields on the gauge group.

κ Minkowski from Fock & Rosly

- In the case of $SO(4,1)$ gauge group the group element can be parameterized as

$$g = \prod_{i=1}^{3} \exp(\theta_i M_i) \prod_{i=1}^{3} \exp(\zeta_i N_i) \prod_{\mu=0}^{3} \exp(x^\mu P_\mu)$$

- One finds

$$\{x^0, x^i\} = -\frac{1}{\kappa} x^i + O(\Lambda, \theta, \zeta), \quad \{x^i, x^j\} = O(\Lambda, \theta, \zeta)$$

- Thus one finds that positions form the κ-Minkowski noncommutative space, which leads to curved space of momenta, κ-Poincare algebra of spacetime symmetries – DSR in short.

κ Minkowski from Fock & Rosly

- In the case of $SO(4,1)$ gauge group the group element can be parameterized as

$$g = \prod_{i=1}^{3} \exp(\theta_i M_i) \prod_{i=1}^{3} \exp(\zeta_i N_i) \prod_{\mu=0}^{3} \exp(x^\mu P_\mu)$$

- One finds

$$\{x^0, x^i\} = -\frac{1}{\kappa} x^i + O(\Lambda, \theta, \zeta), \quad \{x^i, x^j\} = O(\Lambda, \theta, \zeta)$$

- Thus one finds that positions form the κ-Minkowski noncommutative space, which leads to curved space of momenta, κ-Poincare algebra of spacetime symmetries – DSR in short.

κ Minkowski from Fock & Rosly

- In the case of $SO(4,1)$ gauge group the group element can be parameterized as

$$g = \prod_{i=1}^{3} \exp(\theta_i M_i) \prod_{i=1}^{3} \exp(\zeta_i N_i) \prod_{\mu=0}^{3} \exp(x^\mu P_\mu)$$

- One finds

$$\{x^0, x^i\} = -\frac{1}{\kappa} x^i + O(\Lambda, \theta, \zeta), \quad \{x^i, x^j\} = O(\Lambda, \theta, \zeta)$$

- Thus one finds that positions form the κ-Minkowski noncommutative space, which leads to curved space of momenta, κ-Poincare algebra of spacetime symmetries – DSR in short.