Particles and strings in BF theory and quantum gravity

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Zakopane - March 7, 2008

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  - World-line representation : consider the coupling of QFT's Feynman diagrams to QG
- In the second approach the main ingredient is the coupling of particles to the gravitational field (matter $\equiv$ local torsion and curvature defects)
- This approach has led to conceptual advances in 3d and needs to be generalised to higher dimensions
- One possible road is to start by studying curvature defects in the generalisation of 3d gravity to higher dimensions, i.e BF theory

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   - Symmetries

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   - Canonical quantisation
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4. Other approaches - Status report

BF theory

- Let $P = (M, G)$ be a principal bundle over a d-dimensional manifold M with structure group G such that $Lie(G) := \mathfrak{g}$ admits a non-degenerate, Ad-invariant bilinear form tr. The action for BF theory is

$$S_{BF}(A, B) = \int_M \mathrm{tr}\, B \wedge F_A$$

where B is an Ad_P-valued $(d-2)$-form and $F_A = dA + [A \wedge A]$ is the curvature of a connection A on P.

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- The equations of motion are those of a topological field theory

$$F_A = 0, \quad d_A B = 0$$

BF theory

- The action is invariant under the vertical automorphisms of P (ordinary gauge transformations), infinitesimally given by

$$\begin{aligned} \delta_\alpha A &= d_A \alpha \\ \delta_\alpha B &= [B, \alpha] \end{aligned}$$

together with the (reducible) topological symmetry

$$\begin{aligned} \delta_\eta A &= 0 \\ \delta_\eta B &= d_A \eta \end{aligned}$$

forall $(\alpha, \eta) \in \Omega^0(M, Ad_P) \rtimes_{Ad} \Omega^{d-3}(M, Ad_P)$.

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  Einstein's equations in presence of a particle (m, s) propagating along a 1d sub-manifold (world-line) $\gamma \subset M$ yield
  [Deser, Jackiw, 't Hooft - 1984; de Sousa Gerbert - 1990]

$$T = sv\delta_\gamma, \quad F_A = mv\delta_\gamma$$

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  where $v = J_0$ is a fixed unit vector in $\mathfrak{so}(\eta)$.
- This motion can be obtained by varying S_{GR} augmented with the interaction term

$$S_{int}(e, A) = \int_\gamma \mathrm{tr}[(me + sA)v]$$

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  $\Rightarrow$ Perform a gauge transformation and let the gauge parameters $(q, \lambda) \in \Omega^0(\gamma, \mathbb{R}^3) \rtimes_{Ad} \Omega^0(\gamma, \mathrm{SO}(\eta))$ become dynamical degrees of freedom on γ [Freidel, Louapre - 2004]:

$$S_{int}(e, A; q, \lambda) = m \int_\gamma \mathrm{tr}((e + d_A q)\lambda v \lambda^{-1}) + s \int_\gamma \mathrm{tr}(v \lambda^{-1} d_A \lambda)$$

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  This is the action of a massive spinning particle minimally coupled to a Poincaré gauge theory.

  [Balachandran - 1985; de Sousa Gerbert - 1990]

What is the action of a massive spinning particle?

- Configuration space : (connected) Poincaré group manifold

$$\mathcal{P} = I\mathrm{SO}(\eta) = \{(q, \lambda) \mid q \in \mathbb{R}^3, \lambda \in \mathrm{SO}(\eta)\}$$

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$\rightarrow$ q is the position of the particle
$\rightarrow$ λ is related to the momentum and spin vectors p and S

$$\begin{aligned} p &= m\, Ad_\lambda(J_0) \quad (\Rightarrow\ p^2 = \pm m^2) \\ S &= s\, Ad_\lambda(J_0) \quad (\Rightarrow\ S^2 = \pm s^2) \end{aligned}$$

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- The action, for a worldline $\gamma \subset \mathcal{P}$, yields [Balachandran et al. - 1985]

$$S_p(q, \lambda) = \int_\gamma p.dq + \mathrm{str}(J_0 \lambda^{-1} d\lambda)$$

which is the action of a particle coupled to gravity evaluated on $(e, A) = (0, 0)$

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  $\longrightarrow$ What about Plebanski ? (No tetrad dependence)
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  $\longrightarrow$ Follow the lessons from 3d (matter $\equiv$ curvature defects)
- Let $W \subset M$ denote a co-dimension two submanifold of M and consider the interaction term [Baez, Wise, Crans - 06]

$$S_{int}(A, B) = \tau \int_W \mathrm{tr}(Bv)$$

where v is a fixed unit vector in $\mathfrak{g}$ and $\tau \in \mathbb{R}$ is a coupling constant. This leads to the desired equations of motion

$$F_A = \tau v \delta_W$$

BF theory coupled to matter

- Following the same procedure than in 3d, we obtain

[Baez, Perez - 06]

$$S_{int}(A, B; \lambda, q) = \tau \int_W \mathrm{tr}((B + d_A q)\lambda v \lambda^{-1}) \qquad (1)$$

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with λ in $\Omega^0(\Sigma, P)$ and q in $\Omega^{d-3}(\Sigma, Ad_P)$ transforming as

$$\delta_\alpha \lambda = \alpha\lambda \qquad \delta_\eta \lambda = 0$$

and

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In particular, if $d = 4$, W is a surface (world-sheet) and matter is string-like with tension τ

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$$S_{int} = \int_W \mathrm{tr}((B\,\mathcal{E}^2 + d_A q)\lambda v \lambda^{-1}) + \mathcal{E}.\mathcal{F}$$

where $\mathcal{E}$ and $\mathcal{F}$ are respectively the YM electric field and curvature

Canonical quantisation

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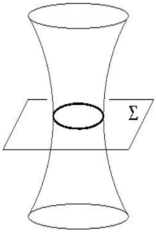
→ Canonical analysis in d=4 :

$M = \mathbb{R} \times \Sigma, \quad \mathscr{S} = \Sigma \cap W$

$$G_i := \epsilon^{abc} D_a B_{bci} + \int_{\mathscr{S}} \dot{x}^a [q_a, p]_i \delta_{\mathscr{S}} \approx 0$$

$$C_i^a := \epsilon^{abc} F_{bci} - \int_{\mathscr{S}} \dot{x}^a p_i \delta_{\mathscr{S}} \approx 0$$



String spin networks

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  $\rightarrow \mathcal{H}_{kin} = \mathbb{C}\{\Psi_\Gamma\}_\Gamma$
  $\Psi_\Gamma \equiv$ string spin networks (SSN) [Thiemann - 97; Baez, Perez - 06]
  SSN : Open graph Γ - finite set of points X on $\mathscr{S}$

$$\Psi_{\Gamma,X}[A,\lambda] = \left[\bigotimes_{e\in\Gamma} \rho_e[g_e] \bigotimes_{x\in X} \rho_x[\lambda_x]\right] \cdot \bigotimes_{v\in\Gamma} \iota_v$$

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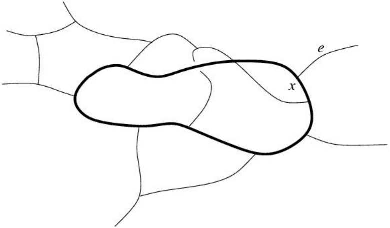

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    $\eta_{\text{phys}} : \text{Cyl} \subset \mathcal{H}_{kin} \longrightarrow \text{Cyl}^*;\ \Psi \mapsto \delta(\hat{C})\Psi$
    $\Rightarrow\ \eta_{\text{phys}}(\text{Cyl}) = \text{Cyl}^*_{\text{phys}} \subset \text{Cyl}^*$ vector space of solutions to the curvature constraint

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- Regularisation: Make sense of $\delta(\hat{C})$

$$\delta(\hat{C}) = \prod_{x \in \Sigma} \delta(\hat{C}(x)) = \int_{\mathcal{N}} \mathcal{D}N \, \exp\left(i \int_{\Sigma} \mathrm{tr}(N \wedge \hat{C}) \right)$$

String/particle duality

- Interesting duality :

$$\int_\Sigma \mathrm{tr}(N \wedge C) = \int_\Sigma \mathrm{tr}(N \wedge F) + \int_{\mathscr{S}} \mathrm{tr}(Np)$$
$$= S^{3d}_{\mathrm{BF+particle}}$$

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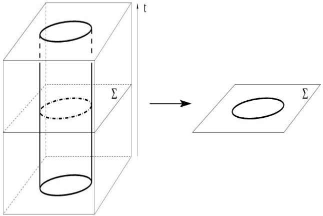
- We have furthermore ($d > 3$):

$$\boxed{(\Omega, \Omega)^d_{\mathrm{BF}+(d-3)-\mathrm{branes}} = \mathcal{Z}^{d-1}_{\mathrm{BF}+(d-4)-\mathrm{branes}}}$$

where $\mathcal{Z}^{d-1} \equiv$ path integral of $(d-1)$-dimensional BF theory coupled to branes



Spinfoam models

- Spinfoam quantisation in 3d [Freidel, Louapre - 2004]

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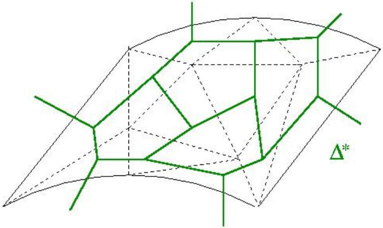
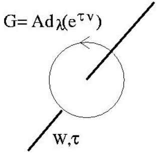
- 4d spinfoam model :

- Fix a triangulation Δ of M and work with the dual two-complex $\Delta^* = (v, e, f)$
- Assign a group element g_e to the edges e of Δ^*
- Holonomies $G_f = \prod_{e \in \partial f} g_e$ around the faces f of Δ^* measure the curvature.



Spinfoam models

- Impose the discretised connection to be flat everywhere except at the location of the brane's world-sheet $W \subset \Delta$



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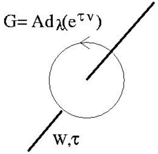

$$\mathcal{Z}_M(W) = \prod_e \int_G dg_e \prod_{f \notin W} \delta(G_f) \prod_{f \in W} \delta_\tau(G_f)$$

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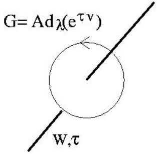

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- The state sum yields

$$\mathcal{Z}_M(W) = \sum_\rho \prod_{f \notin W} \dim \rho_f \prod_{f \in W} \chi_{\rho_f}(e^{\tau v}) \prod_v \{15j\}(\rho)$$

Matter coupling in SF models : where do we stand ?

Approach	3d	4d			
		BF	Plebanski		McD/M
			BC	NM	
Simplicial	Fermions, WF; YM, Speziale	?	YM Oriti,Pfeiffer Mikovic	?	?
Worldline	Particles/Fields Freidel,Louapre Freidel,Livine Barrett Freidel,Baratin	Strings WF,Perez	?	?	?
Sugra	N=1, Oeckl,Livine; N=2, Livine,Ryan	?	?	?	?
Intrinsic	?	?	SM Crane Mikovic	?	?

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and

$$S_{\text{D}} = \frac{1}{4} \int_M \left[(\bar{\psi}\, \Sigma \wedge \nabla \psi) - (\nabla \bar{\psi} \wedge \Sigma\, \psi) \right]$$

where $\Sigma := \Sigma[\mathbf{e}] = \mathbf{e} \wedge \mathbf{e}$ is an $\mathfrak{su}(2)$-valued two-form

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- Discretisation : dual two-complex $\Delta^* = (v, e, f)$ of Δ
  - Assign a Lie algebra element e_f to the faces f
  - Assign a group element g_e to the edges e
  - Holonomies $G_f = \prod_{e \in \partial f} g_e$ around the faces f $\equiv$ curvature
  - Associate a spinor and a co-spinor ψ_v,$\bar{\psi}_v$ to the vertices v

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$$S_{\text{GR}} = \frac{1}{2\kappa} \sum_{f} \operatorname{tr}\left(\mathbf{e}_f\, G_f\right)$$

and

$$S_{\text{D}} = \frac{1}{8} \sum_{e} (\bar{\psi}_{s(e)}\, D_e\, \psi_{t(e)})$$

where

$$D_e = \Sigma_e\, g_e - g_e\, \Sigma_{e^{-1}}, \quad \text{with} \quad \Sigma_e = \frac{1}{3} \sum_{f, f' \supset e} \mathbf{e}_f \mathbf{e}_{f'}\, \operatorname{sgn}(f, f').$$

is the discretised Dirac operator

Fermionic observables

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$$\langle \mathcal{O}_F \rangle_{\text{GR-D}} = \frac{1}{\mathcal{Z}_\Delta} \left(\prod_f \int_{\mathfrak{g}} d\mathbf{e}_f \right) \left(\prod_e \int_G dg_e \right) \times \left(\int_{\mathcal{G}_\Delta} d\mu(\bar{\psi}_v,\ \psi_v) \right) \mathcal{O}_F \ \exp iS_{\text{GR-D}}$$

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- Idea : integrate out the fermionic degrees of freedom, obtain a purely bosonic QG observable

Fermionic observables

- Expectation value of a fermionic obervable $\mathcal{O}_F$:

$$\langle \mathcal{O}_F \rangle_{\text{GR-D}} = \frac{1}{\mathcal{Z}_\Delta} \left(\prod_f \int_{\mathfrak{g}} d\mathbf{e}_f \right) \left(\prod_e \int_G dg_e \right) \times \left(\int_{\mathcal{G}_\Delta} d\mu(\bar{\psi}_v, \psi_v) \right) \mathcal{O}_F \ \exp iS_{\text{GR-D}}$$

- Idea : integrate out the fermionic degrees of freedom, obtain a purely bosonic QG observable

$$\langle \mathcal{O}_F \rangle_{\text{GR-D}} = \frac{\langle \mathcal{O}_B \rangle_{\text{GR}}}{\langle \det\ D \rangle_{\text{GR}}}$$

where

$$\mathcal{O}_B = \left(\int_{\mathcal{G}_\Delta} d\mu(\overline{\psi}_v, \psi_v) \right) \mathcal{O}_F \ \exp iS_{\text{D}}$$

Lattice Feynman diagram expansion

- Expand $\mathcal{O}_B$ into fermionic paths Γ

$$\mathcal{O}_B = \sum_{\Gamma} A_{\Gamma}(\mathbf{e}_f, g_e)$$

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$$\langle \mathcal{O}_F \rangle_{\text{GR-D}} = \frac{1}{\langle \det D \rangle_{\text{GR}}} \sum_{\Gamma} \tilde{A}_{\Gamma} \quad (2)$$

with

$$\begin{aligned} \tilde{A}_{\Gamma} &= \left(\prod_f \int_{\mathfrak{g}} d\mathbf{e}_f \right) \left(\prod_e \int_G dg_e \right) \\ &\times A_{\Gamma}(\mathbf{e}_f, g_e) \; \exp iS_{\text{GR}} \\ &= \prod_f \sum_{j_f} \dim j_f \prod_v A_{v,\Gamma}(j_f) \end{aligned} \quad (3)$$