Lorentzian LQG vertex for finite Immirzi parameter

J. Engle

Albert-Einstein-Institut, Golm

Work done with E. Livine, R. Pereira and C. Rovelli,

related to work involving L. Freidel, K. Krasnov, and S. Speziale

Outline

1. Introduction/Motivation
2. Classical theory:
   (a) Continuum theory
   (b) Discretized canonical theory
3. Quantization: kinematics
   (a) Hilbert space associated with a 3-slice
   (b) Quantum simplicity constraints
   (c) Isomorphism with LQG states
4. Sum over histories and dynamical vertex amplitude
5. Matching of areas
6. Summary and next steps

Introduction/Motivation

1. Investigation of flipped vertex taught us lessons regarding imposing simplicity. One had isomorphism with LQG states and matching with LQG area spectra.

2. However, in flipped vertex, a questionable $\gamma \to 0$ limit had to be taken.

3. γ should be finite anyway, we want a completely clean match with LQG.

- Thus, we take the lessons from the flipped vertex, and apply them to the case of general finite Immirzi parameter.
- Remarkably, again we have exact matching with LQG kinematics and area operators — even more so in the Lorentzian case which we here present, in that *all* LQG area eigenvalues are reproduced.
- Carlo presented Euclidean case; here I present Lorentzian case.

Classical Theory

GR as a constrained BF theory

Holst-BF action

$$S_{Holst-BF} = \frac{1}{8\pi G} \int \mathrm{tr} \left[B \wedge F + \frac{1}{\gamma} ({}^{\star}B) \wedge F \right] + \text{boundary term}$$

Simplicity constraint:

$$B = {}^{\star}(e \wedge e)$$

Substitution into $S_{Holst-BF}$ yields the *Holst action*:

$$S_{Holst} = \frac{1}{8\pi G} \int \mathrm{tr} \left[{}^{\star}(e \wedge e) \wedge F + \frac{1}{\gamma} e \wedge e \wedge F \right] + \text{boundary term}$$

The discretization

Regge triangulation

Introduce triangulation Δ of space-time $\mathcal{M}$ by (oriented) 4-simplices:

triangulation components	(dual to)	symbol
4-simplices	(vertices)	v
tetrahedra	(edges)	t (or e)
triangles	(faces)	f

dual in a 3-slice to
nodes, n
links, l

Basic discrete variables: $V_{vt} \in SL(2,\mathbb{C})$, $B_f(t) \in \mathfrak{so}(3,1)$.

Let $\Lambda : SL(2,\mathbb{C}) \to SO(3,1)$ denote the standard 2-1 homomorphism.

For each triangle f and each pair of tetrahedra $t, t' \in \mathrm{Link}(f)$,

$$U_f(t,t') := V_{tv_1} V_{v_1 t_1} V_{t_1 v_2} \cdots V_{v_n t'}$$

where the product is around the link in the clock-wise direction from t' to t.

Constraints on the variables

1. $U_f(t,t')B_f(t') = B_f(t)U_f(t,t') \qquad \forall\ f$ and $t, t' \in \mathrm{Link}(f)$
2. (closure) $\sum_{f\in t} B_f(t) = 0 \qquad \forall\ t$
3. (discrete simplicity constraints)

$$\text{(i)} \qquad C_{ff} := \frac{1}{4}\mathrm{tr}\left[({}^\star B_f(t))B_f(t)\right] \approx 0 \qquad \forall f$$

(ii) $\exists$ an assignment of a timelike n_t^I to each t, such that

$$C_{ft}^I := n_{tJ}\left({}^\star B_f\right)^{JI} \approx 0 \qquad \forall f \in t$$

(1.) will be imposed prior to varying the action (next slide). **(2.)** will be dictated by the action in quantum theory, **(3i.),(3ii.)** will be imposed separately in quantum theory.

Discrete action

($U_f(t) := U_f(t,t)$, holonomy around the full link, starting at t.)

$$S_{disc} = \frac{1}{16\pi G}\sum_{f\in int\Delta}\mathrm{tr}\left(\left(B_f(t) + \frac{1}{\gamma}{}^\star B_f(t)\right)\Lambda[U_f(t)]\right) + \frac{1}{16\pi G}\sum_{f\in\partial\Delta}\mathrm{tr}\left(\left(B_f(t) + \frac{1}{\gamma}{}^\star B_f(t)\right)\Lambda[U_f(t,t')]\right)$$

Phase space associated with boundary or a 3-slice:

Switch to the dual, 2-complex picture, Δ^*. For each 3-surface Σ intersecting no vertices of Δ^*, let $\gamma_\Sigma := \Sigma \cap \Delta^*$. Use l, n to label links and nodes of γ_Σ.

Basic variables:

$B_l(n) \in \mathfrak{so}(3,1)$, $U_l(n,n') \in SL(2,\mathbb{C})$.

Symplectic structure:

Define array of $\mathfrak{sl}(2,\mathbb{C})$ matrices $\tau^{IJ} = -\tau^{JI}$ by $\tau^{i0} = \frac{1}{2}\sigma^i$, $\tau^{ij} = \frac{-i}{2}\epsilon^{ij}{}_k\sigma^k$

Define $J_l(n) = \dfrac{1}{16\pi G}\left(B_l(n) + \dfrac{1}{\gamma}{}^\star B_l(n)\right)$.

Then

$$
\begin{aligned}
\{J_l(n)^{IJ}, U_l(n,n')\} &= U_l(n,n')\tau^{IJ} \\
\{J_l(n')^{IJ}, U_l(n,n')\} &= \tau^{IJ} U_l(n,n') \\
\{J_l(n)^{IJ}, J_l(n)^{KL}\} &= \lambda^{[IJ][KL]}{}_{[MN]} J_l(n)^{MN}
\end{aligned}
$$

Quantum theory: Kinematics

Hilbert space associated with a 3-slice

Hilbert space associated with Σ:

$$\mathcal{H}_\Sigma = L^2\left(SL(2,\mathbb{C})^{|L(\gamma_\Sigma)|}\right)$$

Let $\hat{J}_l(n)^{IJ}$ denote the right-inv. vect. fields, determined by the basis τ^{IJ} of $\mathfrak{sl}(2,\mathbb{C})$, on the copy of $SL(2,\mathbb{C})$ associated with the link l, with orientation such that the node n is the source of l.

Then

$$\hat{B}_l(n) := 16\pi G\left(\frac{\gamma^2}{\gamma^2+1}\right)\left(\hat{J}_l(n) - \frac{1}{\gamma}{}^\star\hat{J}_l(n)\right)$$

Strategy: to solve cross-simplicity,

1. gauge-fix all n_t^I to $n^I = \delta_0^I$,
2. $SL(2,\mathbb{C})$ gauge-inv. of vertex will then automat. project the states onto the gauge-invariant subspace in the spin-foam sum.

Define $\hat{L}^i_{nl} := \frac{1}{2}\epsilon^i{}_{jk}\hat{J}^{jk}_l(n)$, the generators of the $SU(2)$ rotation subgroup, H, of $SL(2,\mathbb{C})$, preserving $n^I = \delta_0^I$.

Representations of the Lorentz group in the *principal series* are labeled by $N \in \mathbb{Z}$ and $\rho \in \mathbb{R}^+$. Let $\mathcal{H}_{(N,\rho)}$ denote the assoc. carrying space. Decomposing $\mathcal{H}_{(N,\rho)}$ into irreps of H gives

$$\mathcal{H}_{(N,\rho)} = \oplus_{k \geq N/2} \mathcal{H}_k .$$

Generalized $SL(2,\mathbb{C})$ spin-network basis of $\mathcal{H}_\Sigma$

$$\Psi_{\{N_l,\rho_l;k_{nl},i_n\}}(U_l) \equiv \langle U_l|\ \{N_l,\rho_l;k_{nl},i_n\}\rangle := \left(\bigotimes_l D^{(N_l,\rho_l)}(U_l)\cdot\bigotimes_n \left[\left(\otimes_{l\in n}P_{k_{nl}}\right)\otimes i_n\right]\right).$$

where each (N_l,ρ_l) labels a representation of $SL(2,\mathbb{C})$ in the principal series, k_{nl} labels an irrep of $SU(2)$, and i_n is a tensor in $\otimes_{l\in n}\mathcal{H}_{k_{nl}}$.

($\left[\left(\otimes_{l\in n}P_{k_{nl}}\right)\otimes i_n\right]$ is a tensor in $\otimes_{l\in n}\mathcal{H}_{(N_l,\rho_l)}$.)

This basis diagonalizes the operators:

$$\left(\hat{J}_l\cdot\hat{J}_l\right)\Psi_{\{N_l,\rho_l;k_{nl},i_n\}} = \frac{1}{2}(N_l^2-\rho_l^2-4)\Psi_{\{N_l,\rho_l;k_{nl},i_n\}}$$

$$\left(\hat{J}_l\cdot{}^\star\hat{J}_l\right)\Psi_{\{N_l,\rho_l;k_{nl},i_n\}} = N_l\rho_l\Psi_{\{N_l,\rho_l;k_{nl},i_n\}}$$

$$\hat{L}^2_{nl}\Psi_{\{N_l,\rho_l;k_{nl},i_n\}} = k_{nl}(k_{nl}+1)\Psi_{\{N_l,\rho_l;k_{nl},i_n\}}$$

Diagonal quantum simplicity constraints

$$\textbf{(I.)}\quad \hat{\tilde{C}}_{ll} = \frac{1}{4}\epsilon_{IJKL}\hat{B}_l(n)^{IJ}\hat{B}_l(n)^{KL} \propto \left(1-\frac{1}{\gamma^2}\right)\hat{J}_l\cdot{}^\star\hat{J}_l + \frac{2}{\gamma}\hat{J}_l\cdot\hat{J}_l$$
$$= \left(1-\frac{1}{\gamma^2}\right)N\rho + \frac{1}{\gamma}(N^2-\rho^2-4) \approx 0$$

Quantum cross-simplicity constraints

With the $n_t^I = \delta_0^I$ gauge-fixing,

$$\hat{C}^i_{nl} = \left({}^\star\hat{B}_l(n)\right)^{0i} \propto \hat{L}^i_{nl} + \frac{1}{\gamma}\hat{J}_l(n)^{0i} \approx 0$$

As with flipped vertex use master constraint for cross-simplicity:

$$\textbf{(II.)}\quad \hat{\tilde{M}}_{nl} := \sum_i(\hat{C}^i_{nl})^2 = \left(1+\frac{1}{\gamma^2}\right)\hat{L}^2_{nl} - \frac{1}{2\gamma}\hat{J}_l\cdot\hat{J}_l - \frac{1}{2\gamma}\hat{J}_l\cdot{}^\star\hat{J}_l$$

Using (I.), (II.) is equivalent to

$$\textbf{(II'.)}\quad \hat{J}_l\cdot{}^\star\hat{J}_l \approx 4\gamma\hat{L}^2_{nl}$$
$$\Leftrightarrow \quad N_l\rho_l \approx 4\gamma k_{nl}(k_{nl}+1).$$

We also replace (I.) with a constraint with the same classical limit:

$$\textbf{(I'.)}\ \hat{C}_{ll} := \left(1 - \frac{1}{\gamma^2}\right) N_l \rho_l + \frac{1}{\gamma}(N_l^2 - \rho_l^2) \approx 0$$

(I'.) and (II'.) together imply

$$\boxed{k_{ll_-} = \frac{\rho_l}{2\gamma} = \frac{N_l}{2} = k_{ll_+}}$$

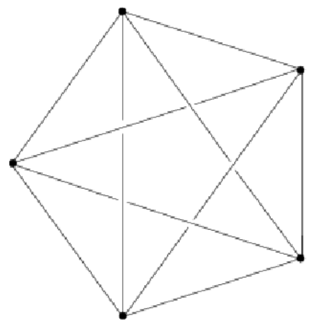
where l_-, l_+ denote the source and target nodes, resp., of l. Let $k_l \equiv k_{ll_-} = k_{ll_+}$. $\{k_l, i_n\}$ then parametrizes solutions of the simplicity constraints; gives obv. isomorphism between $SU(2)$ spin-nets and solutions to simplicity:

$$\boxed{\Psi_{\{k_l,i_n\}} \mapsto \Psi_{\{2k_l,2\gamma k_l;k_l,i_n\}}}$$

History of quantum states: (generalized Lorentzian) spin-foams

A (gen. Lor.) spin-foam is a history of a (gen. Lor.) spin-network. Hence a 2-complex in space-time with faces labelled by representations (N_f, ρ_f), edges labelled by (k_{tf}, i_t).

Vertex amplitude



Find vertex amplitude by evaluating the amplitude for a single 4-simplex v. 10 $B_f(t)$'s and 10 $U_f(t,t')$ are the boundary variables. For each pair $t,t' \in v$, there is a unique f between them, and we write $B_{tt'} = B_f(t)$ and $U_{tt'} = U_f(t,t')$. Recall $J_f(t)^{IJ} = B_f(t) + \frac{1}{\gamma}{}^\star B_f(t)$. Then

$$A[J_{tt'}] = \int dV_{vt}\ e^{i\sum Tr(J_{tt'}\Lambda[V_{tv}V_{vt'}])}.$$

Transforming to the conjugate variables gives

$$\begin{aligned} A[U_{tt'}] &= \int dJ_{tt'} e^{-i\sum Tr(J_{tt'}\Lambda[U_{tt'}])}\ A[J_{tt'}] \\ &= \int dV_{vt} \prod_{tt'} \delta(U_{tt'}V_{t'v}V_{vt}). \end{aligned}$$

This is the amplitude. We transform back to the spin network basis, using the $SL(2,\mathbb{C})$ spin network functions $\Psi_{\{N_f,\rho_f;k_{tf},i_t\}}(U_{tt'})$:

$$
\begin{aligned}
A[N_f, \rho_f; k_{tf}, i_t] &= \int dV_{vt} \Psi^{\text{4-simplex}}_{\{N_f,\rho_f;k_{tf},i_t\}}(V_{tv}V_{vt'}) \\
&= \left(P^{\text{gauge}}_{SL(2,\mathbb{C})} \Psi^{\text{4-simplex}}_{\{N_f,\rho_f;k_{tf},i_t\}} \right) |_{\text{Triv. Conn.}} \\
&\equiv 15j_{SL(2,\mathbb{C})}(N_f, \rho_f; I^t(k_{tf}, i_t))
\end{aligned}
$$

where $I^t(k_{tf}, i_t) := P^t_{SL(2,\mathbb{C})} \left[\left(\otimes_{f \in t} P_{k_{ft}} \right) \otimes i_t \right]$.

Combining with the simplicity constraints, and using the isomorphism of solutions with LQG spin-nets, we obtain the $SU(2)$ LQG spin-foam model with partition function

$$
Z_{\text{GR}} = \sum_{j_f, i_e} \prod_f (2j_f)^2 (1 + \gamma^2) \prod_v A(j_f, i_e)
$$

with $j_f \in \mathbb{N}/2$ and

$$
A(j_f, i_e) = 15j_{SL(2,\mathbb{C})}(2j_f, 2\gamma j_f; I^t(j_f, i_e))
$$

Matching of areas

Classically

$$A^2_{f,4} = \frac{1}{2}({}^\star B)^{IJ}({}^\star B)_{IJ}$$

Using the gauge-fixed cross-simplicity constraint $({}^\star B)_{0i} = 0$, we define $SU(2)$ gauge-fixed area:

$$A^2_{f,3} = \frac{1}{2}({}^\star B)^{ij}({}^\star B)_{ij}$$

Quantization:

$$\widehat{A^2}_{f,3} := \frac{1}{2}({}^\star \hat{B})^{ij}({}^\star \hat{B})_{ij}$$

Spectrum of $\hat{A}_{f,3} = \sqrt{\widehat{A^2}_{f,3}}$ is

$$8\pi G\gamma\sqrt{k_f(k_f+1)}, \quad k_f \in \mathbb{N}/2$$

Exactly as in LQG.

Conclusions

Summary of key points

Regarding the finite γ Lorentzian LQG vertex model:

1. No flip of symplectic structure necessary.
2. Boundary/3-slice Hilbert space isomorphic to that of $SU(2)$ LQG.
3. Exact matching of $SU(2)$-gauge-fixed area spectrum with LQG area spectrum.
4. Works for any $\gamma \in \mathbb{R}^+$ not equal to 1.
5. The above statements are exact with no caveats, in contrast to the Euclidean case, where combinatorics forces restrictions on γ and reduces both the boundary Hilbert space and area spectrum relative to those of LQG.

Some next tasks

1. Result of Reisenberger ensures GFT exists. Explicit construction?
2. Finite amplitude result for these new models, for fixed triangulation?
3. Asymptotics of vertex?
4. Graviton propagator?
5. Relation of vertex to Hamiltonian constraint in LQG?

Relevant new literature

[1] E., Livine, Pereira, and Rovelli 2007 LQG vertex with finite Immirzi parameter, `arXiv:0711.0146`. To be publ. in *Nucl. Phys.* B.

[2] Pereira 2007 Lorentzian LQG vertex amplitude, `arXiv:0710.5043`

[3] E., Pereira 2007 Coherent states, constraint classes, and area operators in the new spin-foam models, `arXiv:0710.5017`

[4] E., Pereira, and Rovelli 2007 The loop-quantum-gravity vertex-amplitude, *Phys. Rev. Lett.* **99** 161301

[5] E., Pereira, and Rovelli 2007 Flipped spinfoam vertex and loop gravity, `arXiv:0708.1236`. To be publ. in *Nucl. Phys.* B.

[6] Livine and Speziale 2007 A new spinfoam vertex for quantum gravity, *Phys. Rev.* D **76** 084028

[7] Freidel and Krasnov 2007 A new spin foam model for 4d gravity, `arXiv:0708.1595`

[8] Livine and Speziale 2007 Consistently solving the simplicity constraints for spinfoam quantum gravity, `arXiv:0708.1915`

GR as a constrained BF theory: more usual formulation, and some more details

Holst-BF action

$$S_{Holst-BF} = \frac{1}{8\pi G}\int \mathrm{tr}\left[B\wedge F + \frac{1}{\gamma}({}^\star B)\wedge F\right]$$

Simplicity constraint:

$$\mathrm{tr}\left[({}^\star B)_{ab}B_{cd}\right] = \mathcal{V}\epsilon_{abcd}$$

where $\mathcal{V} := \frac{1}{4!}\epsilon^{abcd}\mathrm{tr}[({}^\star B)_{ab}B_{cd}]_{abcd}$, with the ϵ's normalized by $\epsilon_{0123} = \epsilon^{0123} = 1$.

Two classes of solutions:

$$B = {}^\star(e\wedge e) \qquad \text{or} \qquad B = e\wedge e$$

Substitution of the first into $S_{Holst-BF}$ yields the *Holst action*:

$$S_{Holst} = \frac{1}{8\pi G}\int \mathrm{tr}\left[{}^\star(e\wedge e)\wedge F + \frac{1}{\gamma}e\wedge e\wedge F\right]$$

The second class of solutions also gives the Holst action, but with $G\gamma$ acting as the Newton constant and γ^{-1} acting as the Barbero-Immirzi parameter.

Note: No topological sector. Just two possible Barbero-Immirzi/Newton constant sectors.

The basic discrete variables

can be introduced and motivated as follows.

1. Introduce a geometry g on $\mathcal{M}$ that is flat on each 4-simplex and is such that all tetrahedra are space-like.

2. In each 4-simplex v, fix a reference point $p(v)$, and in each tetrahedron t, fix a reference point $p(t)$. For each 4-simplex v and tetrahedron t therein, let

$$(V_{vt})^a{}_b : T_{p(t)}\mathcal{M} \to T_{p(v)}\mathcal{M}$$

   denote parallel transport from $p(t)$ to $p(v)$ within v, as determined by g. V_{vt} is unambiguous because g is flat in v.

3. For each v, define a tetrad $e(v)^I_a$ at $p(v)$ such that $g_{ab}|_{p(v)} = e(v)^I_a e(v)_{Ib}$. Likewise for each t, define a tetrad $e(t)^I_a$ at $p(t)$ such that $g_{ab}|_{p(t)} = e(t)^I_a e(t)_{Ib}$.

4. For each v and t therein, define the matrix $(V_{vt})^I{}_J = (V_{tv}^{-1})^I{}_J$ by $e(v)^I_b (V_{vt})^b{}_a = (V_{vt})^I{}_J e(t)^J_a$. The fact that $g_{cd}|_{p(v)}(V_{vt})^c{}_a(V_{vt})^d{}_b = g_{ab}|_{p(t)}$ implies $(V_{vt})^I{}_J \in SO(3,1)$.

5. For each t, because of local flatness of g, $e(t)^I_a$ can be consistently extended to all of t by parallel transport via g. Using this cotetrad field on t, for each triangle f in t, define

$$B_f(t)^{IJ} := \int_f {}^\star(e(t)^I \wedge e(t)^J).$$

Off-diagonal quantum simplicity constraints

These are constraints on the intertwiners. They are second class constraints ($\{C_{f_1 f_2}, C_{f_1 f_3}\} \not\approx 0$), therefore, they should not be imposed strongly ($\hat{C}_{f f'}\Psi = 0$), but weakly in some sense. Inspired by the Gupta-Bleuler formalism, we seek to find a space $\mathcal{H}_{phys}$ such that

$$\langle \phi \mid \hat{C}_{f f'} \mid \psi \rangle = 0 \qquad \forall \phi, \psi \in \mathcal{H}_{phys}.$$

This is not sufficient to determine $\mathcal{H}_{phys}$. We will use our desire to have isomorphism with LQG Hilbert space to guide us, as well as look for an alternative operator equation to use.

The 2-1 homomorphism $\Lambda : SL(2,\mathbb{C}) \to SO(3,1)$:

$$\Lambda(L)^I{}_J = e^I_{AA'} e^{BB'}_J L^A{}_B \overline{L}^{A'}_{B'} \tag{1}$$

where $e^I_{AA'}$ is a fixed "internal soldering form," which, in a once and for all fixed orthonormal basis and spin-frame, may explicitly be taken to be, for example,

$$\begin{aligned} e^{0A}{}_{A'} &= \frac{1}{\sqrt{2}} \mathbb{1} \\ e^{iA}{}_{A'} &= \frac{1}{\sqrt{2}} (\text{Pauli matrices}) \end{aligned}$$

where the $I = 0, i$ index is raised and lowered with the internal Minkowski metric $\eta = \text{diag}(-1, 1, 1, 1)$, and the spinor indicies A, A' are raised and lowered using ϵ_{AB} as in the usual convention: $v^B = v_A \epsilon^{AB}$, $v_A = \epsilon_{AB} v^B$.