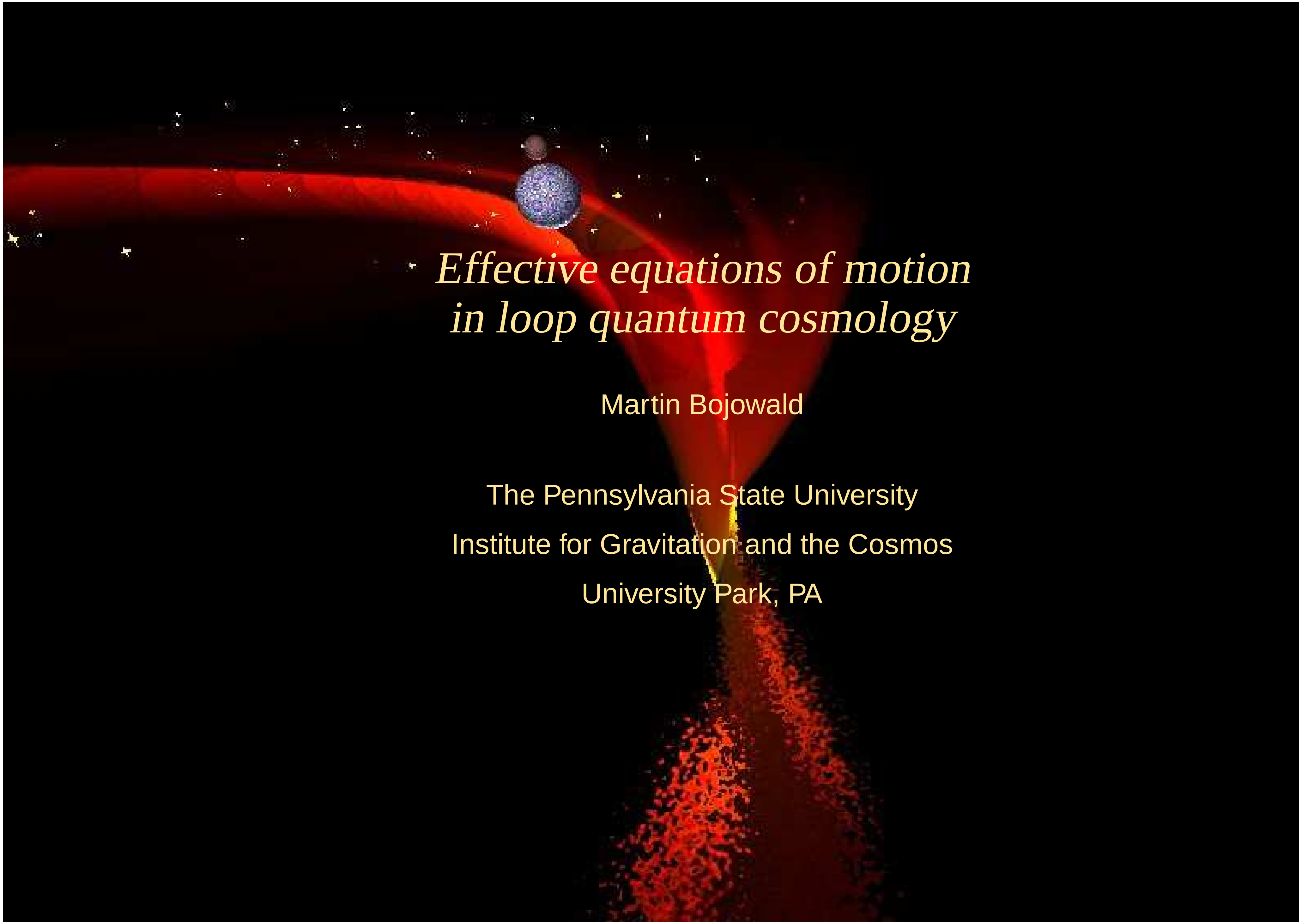

Effective equations of motion in loop quantum cosmology

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Effective equations

Quantum effects can often be analyzed by means of quantum corrections to classical equations, giving rise to observable phenomena.

This requires effective equations, but their purpose is more general:
$\longrightarrow$ systematic checks of robustness of effects,
$\longrightarrow$ self-consistent regimes of applicability (such as semiclassical) via state properties,
$\longrightarrow$ representation independent properties of physical Hilbert spaces,
$\longrightarrow$ conditions for consistent quantizations of constrained systems (anomaly-freedom).

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$\longrightarrow$ conditions for consistent quantizations of constrained systems (anomaly-freedom).

Do not replace complete quantization, but (i) necessary for applications and (ii) provide valuable feedback on full constructions when they are not yet completed.

Ineffective equations

Caution: Not everything called “effective” is truly effective.

In particular: simply replacing classical functions in some way (e.g. $a^{-3} \to d(a)$ or $c \to \sin(\mu c)/\mu$, giving rise to ρ^2-corrections to Friedmann equation) does usually not constitute an effective equation. Those equations can be analyzed and provide some information, but they would be incomplete and potentially misleading.

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There are examples, especially of solvable systems, where such a replacement is OK. But this can be verified only by making sure that they indeed happen to agree with effective equations, or that additional quantum corrections are subdominant in the regimes considered.

Numerical results of isotropic loop quantum cosmology with *free, massless scalar for an unsqueezed Gaussian initial state* confirmed, but new effects emerged in more general settings.

Overview

1. Effective equations in canonical Hamiltonian systems, with some remarks on constraints: representation independent properties of evolving states.
2. Isotropic loop quantum cosmology: a solvable system and perturbation theory.
3. Inhomogeneity and consistent deformations: potential observations and some feedback for the full theory.

For more details, major update of “Loop quantum cosmology” (Liv.Rev.Rel.) is being processed.

Harmonic oscillator

$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$, use *algebra* but no representation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{q}\rangle = \frac{1}{i\hbar}\langle[\hat{q},\hat{H}]\rangle = \frac{1}{m}\langle\hat{p}\rangle \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = \frac{1}{i\hbar}\langle[\hat{p},\hat{H}]\rangle = -m\omega^2\langle\hat{q}\rangle$$

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For $(\Delta O)^2 = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$, $C_{qp} = \frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)^2 &= \frac{\langle[\hat{q}^2,\hat{H}]\rangle}{i\hbar} - 2\langle\hat{q}\rangle\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} = \frac{2}{m}C_{qp}\\
\frac{\mathrm{d}}{\mathrm{d}t}C_{qp} &= -m\omega^2(\Delta q)^2 + \frac{1}{m}(\Delta p)^2\\
\frac{\mathrm{d}}{\mathrm{d}t}(\Delta p)^2 &= -2m\omega^2 C_{qp}
\end{aligned}$$

subject to (generalized) uncertainty relation

$$(\Delta q)^2(\Delta p)^2 - C_{qp}^2 \geq \frac{\hbar^2}{4}$$

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Coherent and squeezed states

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)^2 &= \frac{2}{m}C_{qp} \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}(\Delta p)^2 = -m\omega^2 C_{qp} \\ \frac{\mathrm{d}}{\mathrm{d}t}C_{qp} &= -m\omega^2(\Delta q)^2 + \frac{1}{m}(\Delta p)^2 \end{aligned}$$

allows non-spreading solutions with $C_{qp} = 0$, $\Delta p = m\omega\Delta q$ while following classical trajectory. Satisfies uncertainty relation

$$(\Delta q)^2(\Delta p)^2 - C_{qp}^2 \geq \frac{\hbar^2}{4}$$

if $\Delta q \geq \sqrt{\hbar/2m\omega}$.

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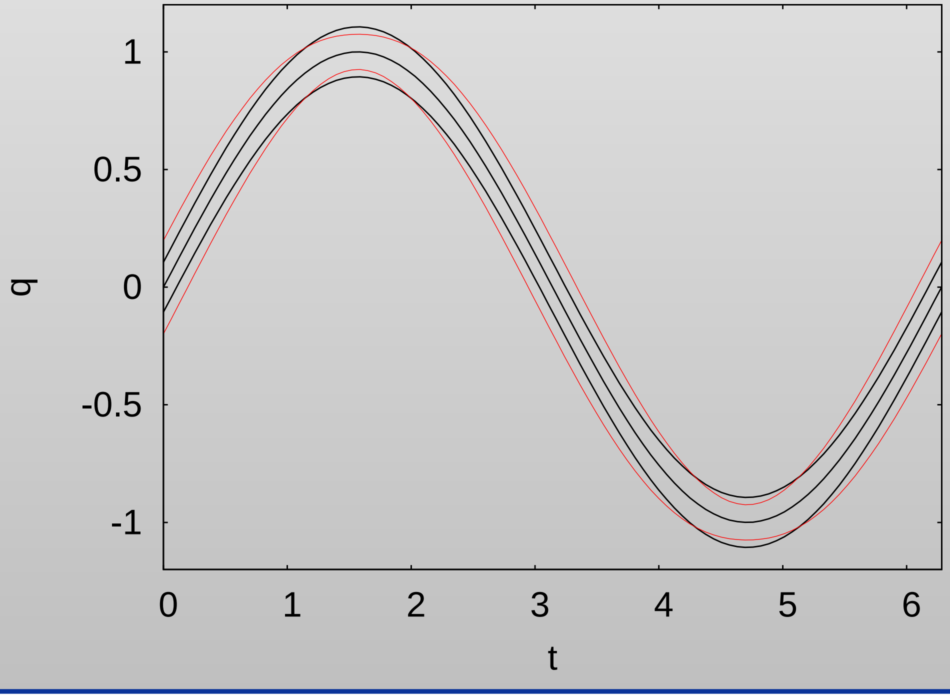

Squeezed states:
spread oscillating if $C_{qp} \neq 0$.
Correlations bounded
for given fluctuations.

Quantum back-reaction

More involved in general, e.g. anharmonic oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{q}) = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \frac{1}{3}\lambda\hat{q}^3$$

Equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{q}\rangle = \frac{1}{m}\langle\hat{p}\rangle \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = -m\omega^2\langle\hat{q}\rangle - \lambda\langle\hat{q}\rangle^2 - \lambda(\Delta q)^2$$

couple expectation values to fluctuation.

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couple expectation values to fluctuation.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)^2 &= \frac{2}{m}C_{qp} \\ \frac{\mathrm{d}}{\mathrm{d}t}C_{qp} &= \frac{1}{m}C_{qp} + m\omega^2(\Delta q)^2 + 6\lambda\langle\hat{q}\rangle(\Delta q)^2 + 3\lambda G^{0,3} \end{aligned}$$

with higher moment $G^{0,3} = \langle(\hat{q} - \langle\hat{q}\rangle)^3\rangle$, and so on …

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Quantum variables

Useful set of coordinates: classical variables $q = \langle\hat{q}\rangle$, $p = \langle\hat{p}\rangle$, and (overcomplete?) *quantum variables* ($n \geq 2$, $0 \leq a \leq n$)

$$G^{a,n} := \langle(\hat{q} - \langle\hat{q}\rangle)^{n-a}(\hat{p} - \langle\hat{p}\rangle)^{a}\rangle_{\rm Weyl}$$

Poisson brackets from commutators $\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} = \langle[\hat{A}, \hat{B}]\rangle/i\hbar$:
$\{q, p\} = 1$, $\{q, G^{a,n}\} = 0 = \{p, G^{a,n}\}$, $\{G^{a,n}, G^{b,m}\} = \cdots$.

[MB, A. Skirzewski: math-ph/0511043, hep-th/0606232]

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Quantum variables dynamical just as classical variables: change in time if *wave packet spreads, deforms and back-reacts* on classical variables which determine the wave packet's peak position.

Exact behavior determined by Schrödinger equation, or by *quantum Hamiltonian* $H_Q := \langle\hat{H}\rangle$ *which couples classical and quantum variables.*

[MB, A. Skirzewski: math-ph/0511043, hep-th/0606232]

Quantum Hamiltonian

Example: an-harmonic oscillator with classical Hamiltonian $H(q,p)=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2+U(q)$; use dimensionless $\tilde{G}^{a,n}=\hbar^{-n/2}(m\omega)^{n/2-a}G^{a,n}$.

Quantum Hamiltonian with *coupling terms*

$$\begin{aligned} H_Q &= \langle H(\hat{q},\hat{p})\rangle = \langle H(q+(\hat{q}-q),p+(\hat{p}-p))\rangle \\ &= \frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2+U(q)+\frac{\hbar\omega}{2}(\tilde{G}^{0,2}+\tilde{G}^{2,2}) \\ &\quad +\sum_{n>2}\frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2}U^{(n)}(q)\tilde{G}^{0,n} \end{aligned}$$

(Formal expansion as shortcut to equations of motion from commutators.)

Equations of motion

H_Q generates Hamiltonian equations of motion $\dot{f} = \{f, H_Q\}$:

$$\begin{aligned}
\dot{q} &= \frac{p}{m} \\
\dot{p} &= -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2} U^{(n+1)}(q)\tilde{G}^{0,n} \\
\dot{\tilde{G}}^{a,n} &= -a\omega\tilde{G}^{a-1,n} + (n-a)\omega\tilde{G}^{a+1,n} - a\frac{U''(q)}{m\omega}\tilde{G}^{a-1,n} \\
&\quad + \frac{\sqrt{\hbar}aU'''(q)}{2(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n-1}\tilde{G}^{0,2} + \frac{\hbar aU''''(q)}{3!(m\omega)^2}\tilde{G}^{a-1,n-1}\tilde{G}^{0,3} \\
&\quad - \frac{a}{2}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2}\tilde{G}^{a-1,n+2}\right) + \cdots
\end{aligned}$$

∞ly many coupled equations for ∞ly many variables.

Low energy effective action

Consistent truncations to finitely many variables possible.

Semiclassical expansion, followed by adiabatic approximation: solve approximately for leading $G^{a,n}$ and insert into equations of motion for q and p.

To first order in $\hbar$, second order in adiabatic approximation:

$$\left(m+\frac{\hbar U'''(q)^2}{32m^2\omega^5\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{5}{2}}}\right)\ddot{q}$$

$$+\frac{\hbar\dot{q}^2\left(4m\omega^2U'''(q)U''''(q)\left(1+\frac{U''(q)}{m\omega^2}\right)-5U'''(q)^3\right)}{128m^3\omega^7\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{7}{2}}}$$

$$+m\omega^2q+U'(q)+\frac{\hbar U'''(q)}{4m\omega\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{1}{2}}}=0\,.$$

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Action principle

Agrees with 1-particle irreducible low energy effective action

$$
\begin{aligned}
\Gamma_{\mathrm{eff}}[q] \quad = \quad & \int \mathrm{d}t \left(\left(m + \frac{\hbar (U''')^2}{32 m^2 (\omega^2 + \frac{U''}{m})^{\frac{5}{2}}} \right) \frac{\dot{q}^2}{2} \right. \\
& \left. -\frac{1}{2} m\omega^2 q^2 - U - \frac{\hbar\omega}{2} \left(1 + \frac{U''}{m\omega^2} \right)^{\frac{1}{2}} \right)
\end{aligned}
$$

to first order in $\hbar$ and second order in adiabatic expansion.

Effective potential visible at first order in $\hbar$ and zeroth order in adiabatic expansion; mass “renormalization” at second order in adiabatic expansion. At higher orders: higher derivative action.

Adiabatic approximation is not necessary as part of general effective equations, may not always be applicable.

Properties

$\longrightarrow$ *Quantum corrections* arise *from back-reaction* of fluctuations and higher moments (loop corrections) unless system is harmonic (free).

$\longrightarrow$ *State properties* such as fluctuations are computed along the way, starting from initial state through initial conditions (interacting vacuum/dynamical coherent states).

$\longrightarrow$ Procedure manageable if *free system* is available as perturbative basis: linear system, $[\hat{J}_i, \hat{H}] = \sum_j a_{ij} \hat{J}_j$ for complete set of basic operators $\hat{J}_i$ such that Ehrenfest equations close.

For canonical basic variables, $\hat{H}$ must be quadratic.
But more possibilities in non-canonical variables (e.g. loop variables).

Properties for constrained systems

[in progress with Barbara Sandhöfer and Artur Tsobanjan]

$\longrightarrow$ Effective constraint by expectation value $C_Q(q,p,G^{a,n}) = \langle\hat{C}\rangle = 0$. But: a single constraint function can remove at most two variables, not the whole tower of quantum variables associated with one classical degree of freedom.

Thus, use *infinitely many constraints* of the form $\langle f(\hat{q},\hat{p})\hat{C}^n\rangle = 0$. Finite subset in approximations, or use techniques as in [Dittrich].

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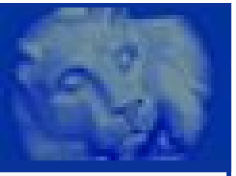

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$\longrightarrow$ *Reality conditions* implementable after imposing constraints, independently of whether zero is in the discrete or continuous part of the spectrum of the constraint operator.

$\longrightarrow$ *Off-shell constraint algebra* can be checked at effective level, and order by order in approximations.

Thus, many issues of specific representations of states, operators and the physical inner product can be avoided.
Yet, key requirements implemented.

Isotropic cosmology

Linear system available for cosmology?

Hamiltonian constraint $c^2\sqrt{|p|} = \frac{4\pi\gamma^2 G}{3}|p|^{-3/2}p_\phi^2$ for free scalar ϕ;

$|p| = a^2$ (triad), $c = \gamma\dot{a}$ (connection, spatial flatness).

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Solving for p_ϕ yields $|p_\phi| \propto |cp| =: |H|$, or $H = VP$ in any canonical pair $V := p^{1-x}$, $P := p^x c$ for $-1/2 < x < 0$.
($x = -1/2$: “$\bar{\mu}$”, $x = 0$: “μ_0”)
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But: *exact free evolution* for any state initially not supported on negative/positive part of $\mathrm{spec}\hat{H}$. Positive/negative $\hat{H}$: analyze left- and right-moving solutions separately.

(Boundary effect at $H = 0$ much less important than potential interaction terms.)

Equations of motion

Quadratic Hamiltonian; *decoupling* of classical and quantum variables.

$$\dot{P} = P \quad , \quad \dot{V} = -V \quad , \quad \dot{G}^{0,2} = 2G^{0,2} \quad , \quad \dot{G}^{1,2} = 0 \quad , \quad \dot{G}^{2,2} = -2G^{2,2}$$

solved by $P(\phi) = c_1 e^{\phi}$, $V(\phi) = c_2 e^{-\phi}$, $G^{0,2}(\phi) = c_3 e^{-2\phi}$, $G^{1,2}(\phi) = c_4$ and $G^{2,2}(\phi) = c_5 e^{2\phi}$. *Singular:* $V = 0$ reached.

Semiclassicality preserved: constant *relative spread* $\Delta V/V = \sqrt{G^{2,2}}/V$.
Uncertainty relation: $c_3 c_5 \geq \hbar^2/4 + c_4^2$.
When saturated, exact solutions for expectation values and fluctuations of dynamical coherent states.

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Loop formulation: Holonomies as basic operators, not connection; difference equation described by $\hat{p}_\phi \psi = |\widehat{V \sin P}|\psi$ in some ordering, non-quadratic, large corrections for $P \sim O(1)$.
(Corrections to inverse volume ignored at this stage.)

Harmonic loop quantum cosmology

Introduce $\hat{J} := \hat{V}\widehat{e^{iP}}$, *linear* $\hat{H} = -\frac{1}{2}i(\hat{J} - \hat{J}^\dagger)$ in specific ordering. Solvable model results. [gr-qc/0608100]

But non-canonical variables $(\hat{V}, \hat{J})$, *centrally extended* $\mathrm{sl}(2,\mathbb{R})$ *algebra*

$$[\hat{V}, \hat{J}] = \hbar\hat{J} \quad , \quad [\hat{V}, \hat{J}^\dagger] = -\hbar\hat{J}^\dagger \quad , \quad [\hat{J}, \hat{J}^\dagger] = -2\hbar\hat{V} - \hbar^2$$

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Equations of motion

$$\dot{V} = -\tfrac{1}{2}(J + \bar{J}) \quad , \quad \dot{J} = -\tfrac{1}{2}(V + \hbar) = \dot{\bar{J}}$$

with general solution

$$\begin{aligned} V(\phi) &= \tfrac{1}{2}(Ae^{-\phi} + Be^{\phi}) - \tfrac{1}{2}\hbar \\ J(\phi) &= \tfrac{1}{2}(Ae^{-\phi} - Be^{\phi}) + iH \end{aligned}$$

“Bounce” since $|V| \to \infty$ for $\phi \to \pm\infty$, but *could enter deep quantum regime* if $AB < 0$; solvable model would break down.

Reality condition

Classical: $J\bar{J} = V^2$. Quantum: $\hat{J}\hat{J}^\dagger = \hat{V}^2$. Related to *physical inner product:* $\exp(iP)$ becomes unitary operator.

Expectation values: $|\langle\hat{J}\rangle|^2 - (\langle\hat{V}\rangle + \frac{1}{2}\hbar)^2 = G^{VV} - G^{J\bar{J}} + \frac{1}{4}\hbar^2$.

Implies $AB = H^2 + O(\hbar)$, *bouncing* solution ($e^{2\delta} = B/A$)

$$V(\phi) = H\cosh(\phi - \delta) \quad , \quad J(\phi) = -H(\sinh(\phi - \delta) - i)$$

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Uncertainties:

$$\begin{aligned} \dot{G}^{VV} &= -G^{VJ} - G^{V\bar{J}} \quad , \quad \dot{G}^{JJ} = -2G^{VJ} \quad , \quad \dot{G}^{\bar{J}\bar{J}} = -2G^{V\bar{J}} \\ \dot{G}^{VJ} &= -\frac{1}{2}G^{JJ} - \frac{1}{2}G^{J\bar{J}} - G^{VV} \, , \, \dot{G}^{V\bar{J}} = -\frac{1}{2}G^{\bar{J}\bar{J}} - \frac{1}{2}G^{J\bar{J}} - G^{VV} \\ \dot{G}^{J\bar{J}} &= -G^{VJ} - G^{V\bar{J}} \end{aligned}$$

$\longrightarrow$ Reality preserved: $\dot{G}^{VV} - \dot{G}^{J\bar{J}} = 0$.

$\longrightarrow$ Near saturation, $(\Delta V)^2 = G^{VV} \approx \hbar H\cosh(2(\phi - \delta_2))$.

In general, $\delta_2 \neq \delta$: asymmetric fluctuations.

Effective Friedmann equation

Linear system: Precise effective Hamiltonian

$p_\phi = \langle \hat{H} \rangle = (2i)^{-1}(J - \bar{J}) = V \sin P$, equation of motion

$$\frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{1}{i\hbar}\langle[\hat{V}, \hat{H}]\rangle = -\frac{J + \bar{J}}{2} = -V \cos P = -V\sqrt{1 - \left(\frac{p_\phi}{V}\right)^2}$$

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Using $\dot{\phi} = V^{-3/2(1-x)} p_\phi$, this implies

$$\dot{V}^2 = \left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \dot{\phi}^2 = p_\phi^2 V^{-(1+2x)/(1-x)} \left(1-\left(\frac{p_\phi}{V}\right)^2\right)$$

and the corrected Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{2(1-x)^2}\rho_{\mathrm{free}}(1-2a^{2+4x}\rho_{\mathrm{free}})$$

for a *free, massless scalar:* evolution of expectation value. Proves correctness of this effective equation [gr-qc/0608100], reproduced recently from exact wave function.

[Ashtekar, Corichi, Singh: 0710.3565]

Fluctuations [gr-qc/0703144]

$$\begin{aligned}
G^{VV}(\phi) &= \frac{1}{2}(c_3 e^{-2\phi} + c_4 e^{2\phi}) - \frac{1}{4}(c_1 + c_2) \\
G^{JJ}(\phi) &= \frac{1}{2}(c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4}(3c_2 - c_1) - i(c_5 e^{\phi} - c_6 e^{-\phi}) \\
G^{\bar{J}\bar{J}}(\phi) &= \frac{1}{2}(c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4}(3c_2 - c_1) + i(c_5 e^{\phi} - c_6 e^{-\phi}) \\
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G^{J\bar{J}}(\phi) &= \frac{1}{2}(c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4}(3c_1 - c_2)
\end{aligned}$$

Dynamical coherent states

Uncertainty relations

$$\begin{aligned} G^{VV}G^{J+\bar{J},J+\bar{J}}-(G^{V,J+\bar{J}})^2 &\geq \hbar^2H^2\\ G^{VV}G^{i(J-\bar{J}),i(J-\bar{J})}-(G^{V,i(J-\bar{J})})^2 &\geq \frac{1}{4}\hbar^2(\langle\hat{J}\rangle+\langle\hat{J}^\dagger\rangle)^2\\ G^{J+\bar{J},J+\bar{J}}G^{i(J-\bar{J}),i(J-\bar{J})}-(G^{J+\bar{J},i(J-\bar{J})})^2 &\geq \hbar^2(2\langle\hat{V}\rangle+\hbar)^2 \end{aligned}$$

Saturation (using $\langle\hat{V}\rangle(\phi)=A\cosh(\phi)-\frac{1}{2}\hbar$) implies:

$$\begin{aligned} 4c_3c_4 &= H^2\hbar^2+\frac{1}{4}(c_1+c_2)^2\\ (c_1-c_2)c_3-c_6^2 &= \frac{1}{4}A^2\hbar^2=(c_1-c_2)c_4-c_5^2\\ 4c_5c_6 &= A^2\hbar^2+c_2^2-c_1^2 \end{aligned}$$

where $(\Delta H)^2=\frac{1}{2}(c_1-c_2)$ and $A^2=H^2+c_1-\frac{1}{4}\hbar^2$ (reality).

Change of fluctuations

Solution:

$$D := \left| \lim_{\phi\to-\infty} \frac{G^{VV}}{\langle\hat{V}\rangle^2} - \lim_{\phi\to\infty} \frac{G^{VV}}{\langle\hat{V}\rangle^2} \right| = 2\frac{|c_3 - c_4|}{A^2}$$

$$= 4\frac{H}{A}\sqrt{\left(1 - \frac{H^2}{A^2} + \frac{1}{4}\frac{\hbar^2}{A^2}\right)\frac{(\Delta H)^2}{A^2} - \frac{1}{4}\frac{\hbar^2}{A^2} + \left(\frac{H^2}{A^2} - 1\right)\frac{(\Delta H)^4}{A^4}}$$

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Compare with [Corichi, Singh: 0710.4543]

$$D \leq 8\beta\epsilon \sim 8\Delta P/P$$

Similar, but here much sharper: right hand side of order relative fluctuation *squared,* not linear.

Example: $\beta\epsilon_+ \approx \alpha_2(\Delta V/V)_+ \approx \sqrt{\alpha_2}10^{-57}$, thus $D \leq \sqrt{\alpha_2}10^{-57}$.

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$(\Delta V/V)_-^2 \leq \sqrt{\alpha_2}10^{-57} + (\Delta V/V)_+^2 \approx \sqrt{\alpha_2}10^{-57}$, thus
$(\Delta V/V)_- \approx \alpha_2^{1/4}10^{-28} \approx \alpha_2^{3/4}10^{28}(\Delta V/V)_+$

Cosmic forgetfulness [Nature Phys. 3 (2007) 523]

No strong restrictions for dynamical coherent states.
Solve for c_4, divide:

$$\begin{aligned} \left|1-\frac{(\Delta V)_-}{(\Delta V)_+}\right| &= \frac{|c_4-c_3|}{c_4} \\ &= \frac{2\frac{H}{A}\delta}{\frac{\delta^2}{2(\Delta H)^2} \pm \frac{H}{A}\delta + \frac{1}{2}\frac{H^2}{A^2}(\Delta H)^2 + \frac{1}{8}\frac{A^2\hbar^2}{(\Delta H)^2}} \end{aligned}$$

where

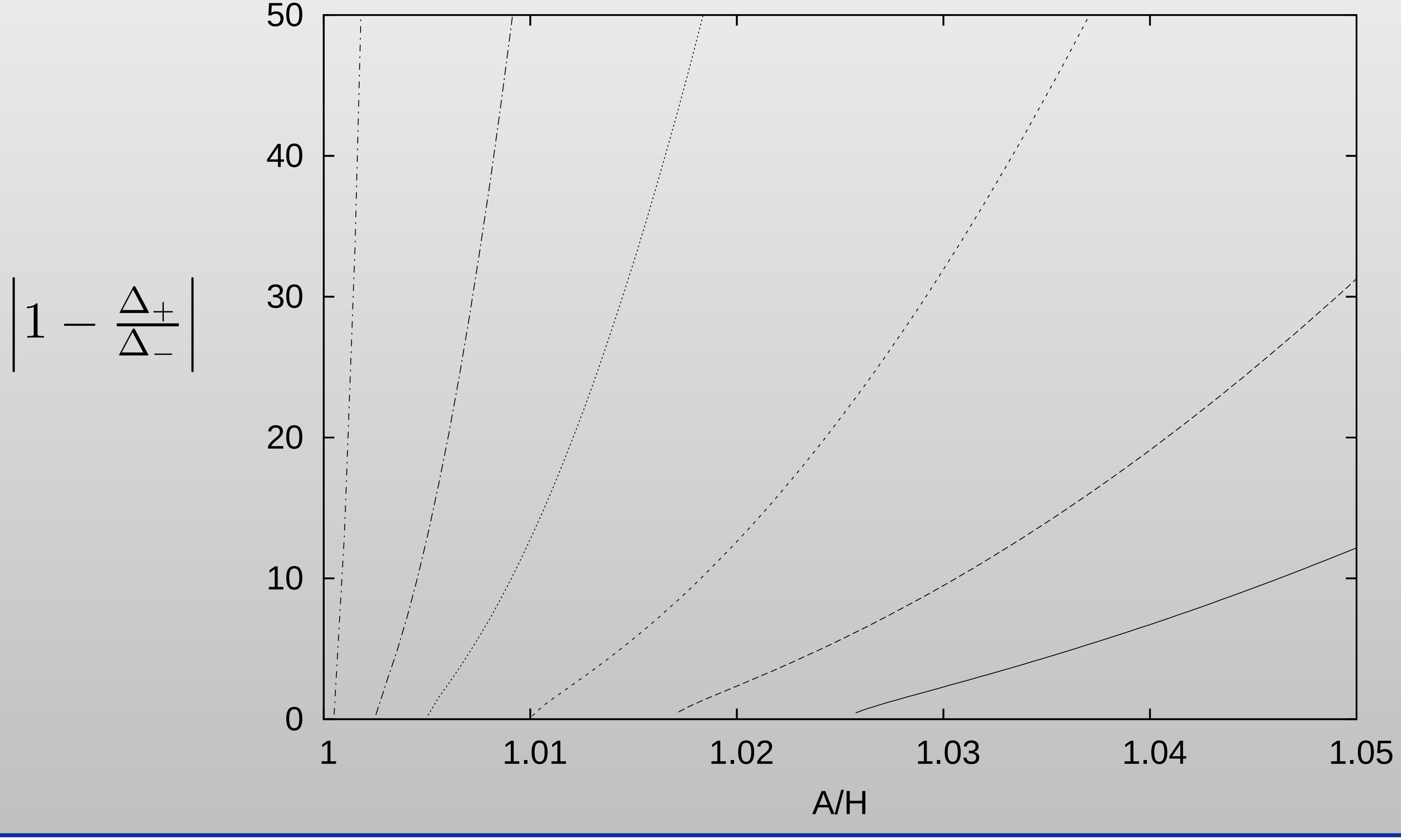
$$4\left(\frac{H^2}{A^2}-1\right)(\Delta H)^4 + 4\left(A^2 - H^2 + \frac{1}{4}\hbar^2\right)(\Delta H)^2 - A^2\hbar^2 =: 4\delta^2$$

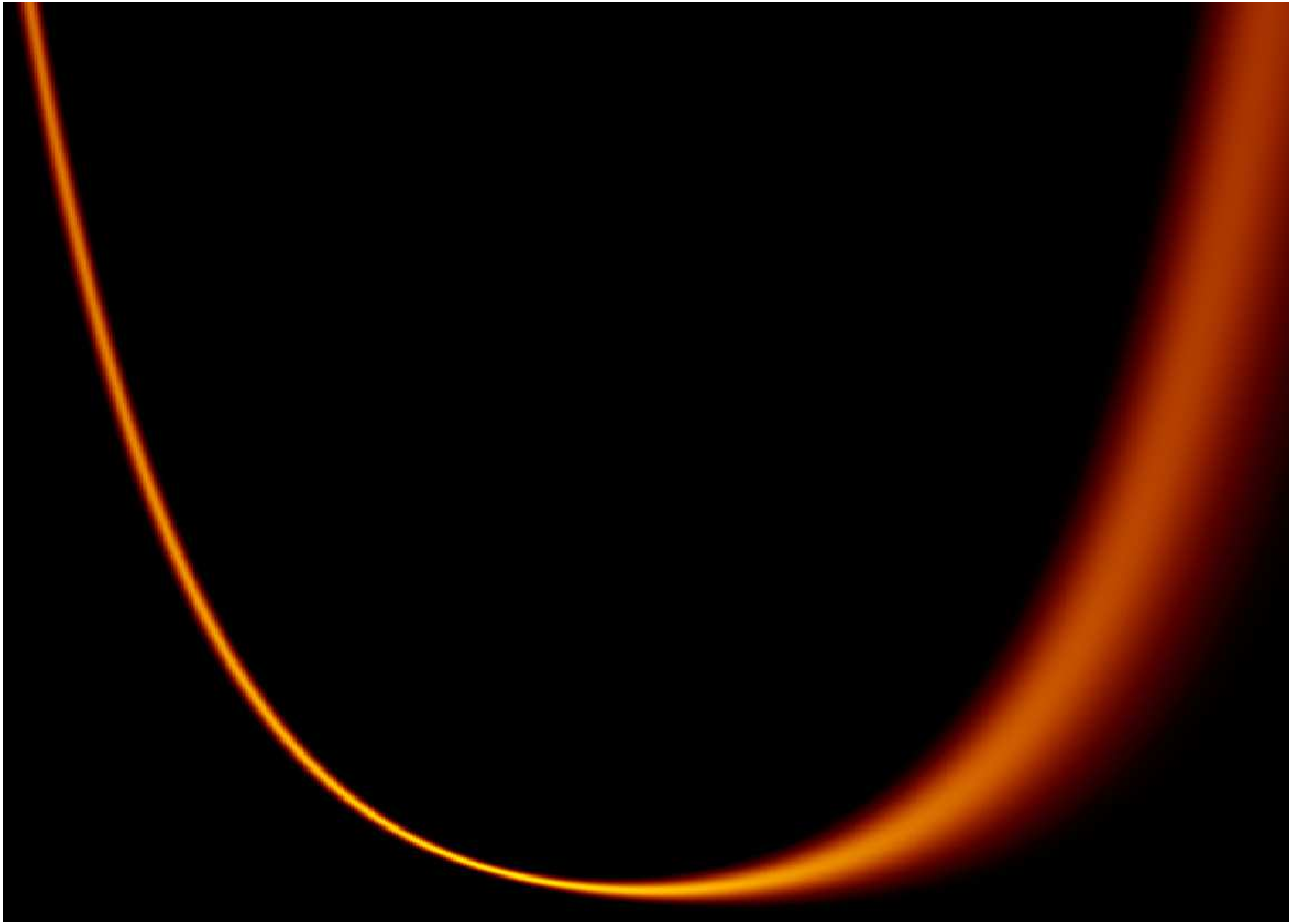
Order of magnitude not small, easily of $O(10)$.
Not strongly restricted even for $c_1 + c_2 = 0$, minimizing c_3c_4 in saturation equation $4c_3c_4 = H^2\hbar^2 + \frac{1}{4}(c_1+c_2)^2$.

Sensitivity [0710.4919]

Asymmetry of fluctuations before and after the big bang extremely sensitive to initial values (A, H, ΔH).

For different values of $H = \langle \hat{H} \rangle$, steepness increasing:





Interactions [MB, Hernández, Skirzewski: 0706.1057]

Free bounce model may replace harmonic oscillator as “free” gravitational theory even with loop quantization. Matter potential, anisotropy or inhomogeneities may be included *perturbatively.*

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Free bounce model may replace harmonic oscillator as “free” gravitational theory even with loop quantization. Matter potential, anisotropy or inhomogeneities may be included *perturbatively.*

Effective equation with non-zero potential:

$$\begin{aligned}
\dot{p} \;=\;& -\frac{J+\bar{J}}{2}+\frac{J+\bar{J}}{(J-\bar{J})^2}p^3V(\phi)\\
&+3\frac{p^3(J+\bar{J})}{(J-\bar{J})^4}(G^{JJ}+G^{\bar{J}\bar{J}}-2G^{J\bar{J}})V(\phi)\\
&-6\frac{p^2(J+\bar{J})}{(J-\bar{J})^3}(G^{pJ}-G^{p\bar{J}})V(\phi)+3\frac{p(J+\bar{J})}{(J-\bar{J})^2}G^{pp}\,V(\phi)\\
&-\frac{2p^3}{(J-\bar{J})^3}(G^{JJ}-G^{\bar{J}\bar{J}})V(\phi)+\frac{3p^2}{(J-\bar{J})^2}(G^{pJ}+G^{p\bar{J}})V(\phi)
\end{aligned}$$

together with equations for $\dot{J}$ and $\dot{G}^{a,2}$ as *independent variables.*

Effective Friedmann equation [0801.4001]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho_Q}{\rho_{\rm crit}}\right) - \left(\frac{4\pi G}{3}\right)^{3/2}\sqrt{1-\frac{\rho_Q}{\rho_{\rm crit}}}\frac{p_\phi W\eta}{a^3} + \left(\frac{4\pi G}{3}\right)^2 W(\phi)^2\eta^2$$

with fluctuation-corrected ρ_Q and correlations η.

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with fluctuation-corrected ρ_Q and correlations η.

Local evolution near semiclassical state not changed much, but correct question: What happens to bounce if an initial semiclassical state *at large volume* is evolved all the way to small volume?

Generically, quantum states (including η), change much during long evolution. Quantum corrections in effective equations crucial; *not enough known to conclude generic presence of bounces even in isotropic models.*

Further models

$\longrightarrow$ *Positive spatial curvature:* Not exactly solvable, thus quantum back-reaction. But only significant around recollapse, where quantum variables behave like

$$\begin{aligned}\dot{G}^{PP} &= \frac{P}{\sqrt{P^2+1}}G^{PP} \quad , \quad \dot{G}^{VP} = -\frac{1}{2}\frac{V}{(P^2+1)^{3/2}}G^{PP} \\ \dot{G}^{VV} &= -\frac{V}{(P^2+1)^{3/2}}G^{VP} - \frac{P}{\sqrt{P^2+1}}G^{VV}\end{aligned}$$

Volume fluctuations change significantly only for squeezed states, $G^{VP} \neq 0$.

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$$\dot{G}^{VV} = -\frac{V}{(P^2+1)^{3/2}}G^{VP} - \frac{P}{\sqrt{P^2+1}}G^{VV}$$

Volume fluctuations change significantly only for squeezed states, $G^{VP} \neq 0$.

$\longrightarrow$ *Positive cosmological constant:* classical volume diverges at finite ϕ, and so do volume fluctuations in perturbation analysis. Here, quantum back-reaction becomes dominant and could radically change the volume expectation value.

Conjecture: Behavior of quantum variables, convergence issues reflect properties of self-adjoint extensions of Hamiltonian.

Reduced variables

Recall: variables $V := p^{1-x}$, $P := p^x c$ for $-1/2 < x < 0$ of solvable model x-dependent, which affects perturbations. In particular, $P(c,p)$ used in “holonomies” and $J = V\exp(iP)$, which become triad dependent.

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Explained by average treatment of *lattice refinement* in inhomogeneous setting: holonomy along straight curve of length ℓ_0 evaluated in isotropic connection $\tilde{c}$:

$$h = \exp(i\ell_0\tilde{c}) = \exp(i\ell_0 c/V_0^{1/3}) = \exp(ic/\mathcal{N}^{1/3})$$

where $\mathcal{N}$ is the number of vertices in a cell of (coordinate) volume V_0. Reduction done for fixed lattice, but average treatment allows for dependence of $\mathcal{N}$ on the total spatial volume. (Inverse corrections enlarged from refined fluxes.)

$\longrightarrow$ No dependence on V_0 (“gauge”) for any x.

Consequences

$\longrightarrow$ Triad dependent holonomies, where $\mathcal{N} \propto V^k$ with $0 < k < 1$ implies $-1/2 < x < 0$ in $P = p^x c$.

Examples: • For $x = 0$ ("μ_0"), no refinement since $\mathcal{N} = \mathrm{const}$. Not suitable for all semiclassical regimes, mainly in presence of a positive cosmological constant. Also: bounce can happen too early due to an unrefined discrete quantum gravity state.

• For $x = -1/2$ ("$\bar{\mu}$"): $\mathcal{N} \propto V$, discreteness scale preserved even during expansion. Several other phenomenological considerations point to $x \approx -1/2$, but may be larger.

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• Anisotropy: Different possibilities for direction dependence of refinement. New feature: Not necessarily equidistant difference equation. [MB, Cartin, Khanna]
Appears necessary for good behavior with positive spatial curvature. Even $\mathcal{N} \propto V$ ("$\bar{\mu}$") seems problematic e.g. near the horizon of Kantowski–Sachs models. [Böhmer, Vandersloot; Chiou]

Covariance?

Dynamics changes, classical symmetries may not be preserved.
Equations have to satisfy consistency requirements:
Anomaly freedom in space-time gauge symmetries becomes non-trivial in inhomogeneous situations.

Direct treatment:

Quantum commutator $[\hat{H}[N], \hat{H}[M]]$ of correct form?

Which semiclassical state for effective Hamiltonian $\langle\hat{H}\rangle$?

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Which semiclassical state for effective Hamiltonian $\langle \hat{H} \rangle$?

Canonical effective procedure:

First compute (or parameterize) effective constraints, then check *effective constraint algebra.*

Properties of *dynamical semiclassical state* determined order by order in expansion (just like interacting vacuum in perturbative quantum field theory).

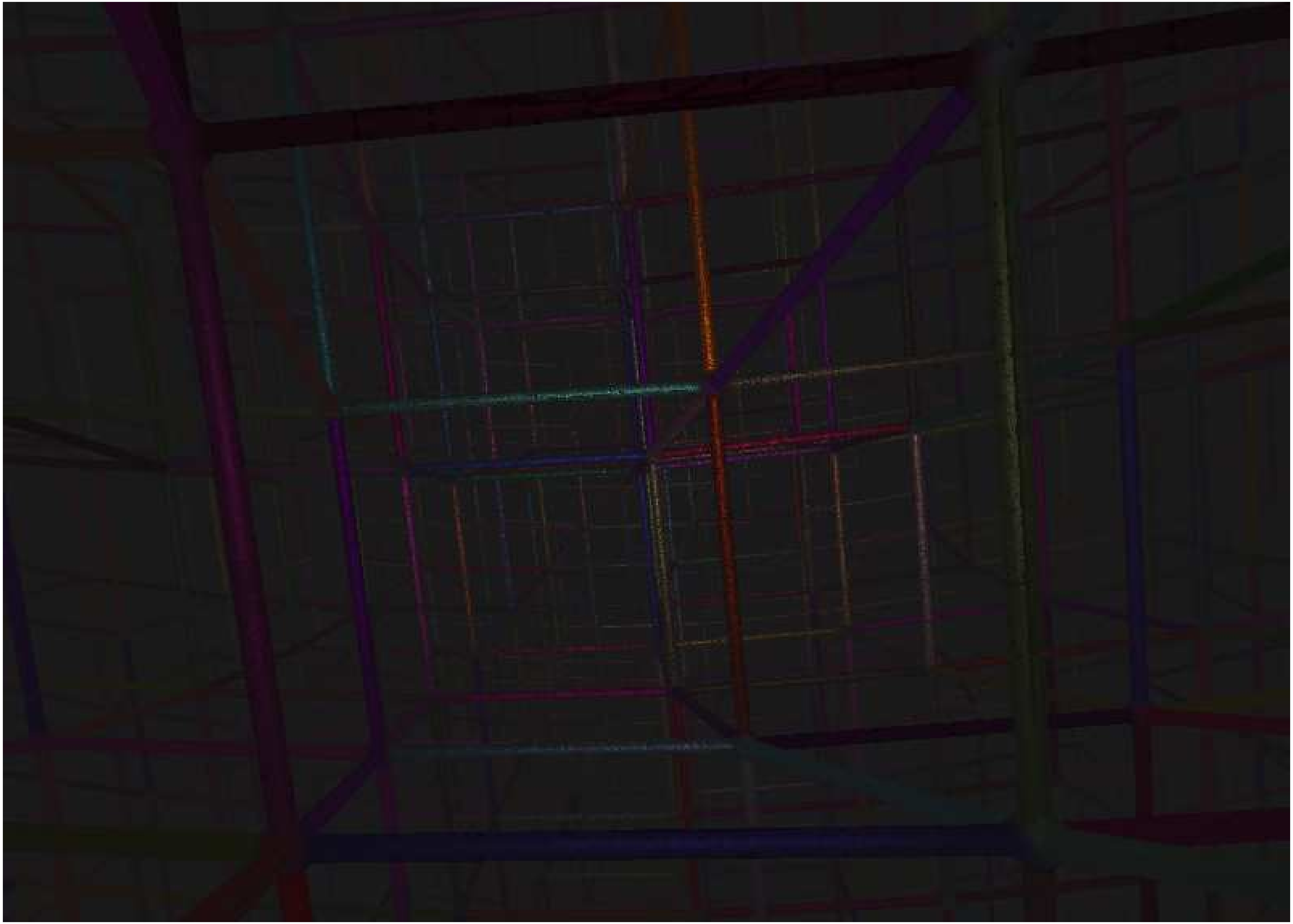
Current results

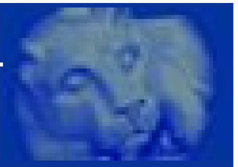
[In progress with Golam Hossain, Mikhail Kagan, Juan Reyes, . . .]

Focus so far mostly on corrections from inverse triad components, rather than holonomies or quantum back-reaction.

Consistent deformations found in several cases, including spherical symmetry and linear cosmological perturbations around spatially flat Friedmann–Robertson–Walker.

Most non-trivial: scalar modes. Equations of motion for gauge-invariant perturbations containing quantum corrections found such that:
$\longrightarrow$ Result from first class constraint algebra, but with *corrections to the classical algebra.*
$\longrightarrow$ Requires *off-shell closure* of constraint algebra, realized to linear order in effective constraints.
$\longrightarrow$ Quantum corrections computed (in incomplete derivation) from quantum constraints C_Q, but *kinematical coherent states not sufficient.*





Gravitational waves [MB, G. Hossain: 0709.2365]

Example: Propagation of gravitational waves compared to light.
Hamiltonian (non-trivial correction α from inverse triad)

$$H_{\mathrm{G}} = \frac{1}{16\pi G}\int_{\Sigma} \mathrm{d}^3x\,\alpha(E^a_i)\,\frac{E^c_jE^d_k}{\sqrt{|\det E|}}\left(\epsilon_i{}^{jk}F^i_{cd} - 2(1+\gamma^2)K^j_{[c}K^k_{d]}\right)$$

implies linearized wave equation

$$\frac{1}{2}\left[\frac{1}{\alpha}\ddot{h}^i_a + 2\frac{\dot{a}}{a}\left(1 - \frac{2a\,\mathrm{d}\alpha/\mathrm{d}a}{\alpha}\right)\dot{h}^i_a - \alpha\nabla^2 h^i_a\right] = 8\pi G\Pi^i_a$$

for tensor mode h^i_a on cosmological background with scale factor a and source-term Π^i_a.
Dispersion relation for gravitational waves:

$$\omega^2 = \alpha^2 k^2$$

$\alpha > 1$ (for $a > a_*$) from perturbative corrections: super-luminal?

Causality

Compare with electrodynamics, Hamiltonian:

$$H_{\mathrm{EM}} = \int_\Sigma \mathrm{d}^3x \left[\alpha_{\mathrm{EM}}(q_{cd}) \frac{2\pi}{\sqrt{q}} E^a E^b q_{ab} + \beta_{\mathrm{EM}}(q_{cd}) \frac{\sqrt{q}}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd}\right]$$

wave equation

$$\partial_t \left(\alpha_{\mathrm{EM}}^{-1} \partial_t A_a\right) - \beta_{\mathrm{EM}} \nabla^2 A_a = 0$$

dispersion relation

$$\omega^2 = \alpha_{\mathrm{EM}} \beta_{\mathrm{EM}} k^2$$

Also “super-luminal” compared to classical speed of light.

Anomaly-freedom:

$$\alpha^2 = \alpha_{\mathrm{EM}} \beta_{\mathrm{EM}}$$

from effective constraint algebra, physically not super-luminal.

Conclusions

Effective equations can be systematically derived in canonical formulation, provided a free solvable system is available as in loop quantum cosmology. Provides *exact effective equations* as well as basis for *perturbative effective equations* in more complicated systems. *Handle on physical Hilbert space.*

Control over *dynamical coherent states*. Generically, fluctuations before and after bounce unrelated due to squeezing.

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Interactions: *additional correction terms.* Small as long as state semiclassical, but become important as state develops stronger quantum features. Especially important for cosmology.

Reduction of basic operators from full holonomy-flux algebra gives rise to different models depending on *lattice refinement.* Too narrow focus (initially: μ_0, more recently: $\bar{\mu}$) may be dangerous. Precise form already constrained in models, thus providing *feedback for full theory.*